Complex Price Dynamics in Vertically Linked Cobweb Markets in Less Developed Countries

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Abstract

Advances in nonlinear dynamics have resuscitated interest in the theory of endogenous price fluctuations. In this paper, we enrich the traditional cobweb model by incorporating vertical linkages between agricultural markets, an important aspect of agricultural production in less developed countries that has been largely ignored by the existing literature on chaotic cobweb models. To this end, under sufficiently general cost and demand functions, we develop a parsimonious, dynamic model to capture mutual interdependencies between upstream and downstream markets arising from the vertically linked structure of agricultural values chains. The model is solved under the naïve expectations hypothesis to derive a system of coupled, time-delay difference equations characterizing price dynamics in vertically linked cobweb markets. Mathematical and numerical analyses reveal that vertical linkages between downstream and upstream cobweb markets have profound effects on local stability, global price dynamics and onset of chaos. We also add an empirical dimension to an otherwise theoretical literature by benchmarking moments of prices simulated by our model against actual poultry value chain price data from Pakistan. Simulated prices reproduce the stylized patterns observed in the actual price data, i.e., quasi-periodic behavior, positive first-order autocorrelation, relatively higher variation in upstream prices compared to downstream prices, low skewness and negative kurtosis. In doing so the paper addresses a major criticism of the theory of endogenous price fluctuations, most notably the failure of chaotic cobweb models to adequately replicate positively autocorrelated prices. From a policy perspective, our model predicts that the increasingly unpredictable behavior of agricultural commodity prices in less developed countries may be driven by improvements in technological efficiency at the downstream level, combined with declines in consumer’s sensitivity to agricultural commodity prices.

1. Introduction

Fluctuations in agricultural product prices can have a profound impact on the welfare of consumers and farmers, especially in countries plagued by heavy dependence on domestic production of food-based commodities, institutional voids, and imperfect financial markets. The cobweb model (Ezekiel, 1938 and Nerlove, 1958) has frequently been used as a conceptual framework to understand cyclical fluctuations in the prices of non-storable agricultural commodities. The mechanics of the cobweb model are fairly straightforward and intuitive. Due to the biological nature of agricultural production, there is a fixed time-lag between the decision to produce and the realization of production, and farmers are assumed to form expectations of future prices on current prices (i.e., naïve expectations, or, more generally, static expectations). Consequently, farmers increase planned production when current prices are high and vice versa, leading to either a supply glut or a shortage at the end of the production cycle. Moreover, in the absence of storage, imbalances between supply and demand are further amplified by the relatively inelastic demand for agricultural commodities, resulting in large, cyclical price fluctuations over time.

While simplistic, such a view of agricultural markets is not unrealistic. A delayed supply response is intrinsic to agricultural production and, in the absence of adequate marketing infrastructure, commodity storage plays a negligible role in less developed countries. Although contentious, the literature does lend support to the naïve expectations hypothesis. In their seminal work on bounded rationality, Brock and Hommes (1997) prove that forecasts based on naïve expectations are potentially rational if information acquisition and processing is costly. Intuitively, if the cost of acquiring and processing information exceeds the perceived benefits of "better" forecasts, then optimizing farmers will choose not to employ the available information and instead rely on "cheap" naïve forecasts[1]. The empirical literature also suggests that farmers rely on simple strategies to forecast prices, and production decisions are often based on current market prices. For example, Chavas (1999a) finds that the price expectations formed by approximately 90% of US poultry farmers are consistent with the naïve expectations hypothesis. Baak (1999) and Chavas (1999b and 2000) find that the majority of beef and pork farmers in the USA behave naively. This type of boundedly rational behavior has also been documented in laboratory experiments (Hommes et al., 2007; Sonnemans et al., 2004). For instance, Heemeijer et al., (2009) observe that subjects prefer to use simple rules of thumb to forecast future prices that are often inconsistent with the rational expectation hypothesis.

Nevertheless, in the framework of neoclassical economics, models are ultimately judged upon their predictive ability and not on the basis of their underlying assumptions (Lucas 1980). Few would argue that the price dynamics generated by traditional cobweb models are realistic. With linear supply and demand functions, the range of price dynamics is limited to convergence to a

[1] Timely and accurate information on breeding stocks, inventories and production is simply not available in most countries. As a result, information acquisition costs can be very high. Thus, from a practical standpoint, one may ask what types of information are usually available to farmers? Current market prices are perhaps the only source of reliable information available to farmers at the time of making production decisions, especially in less developed countries plagued by numerous institutional voids.

unique steady-state and explosive, damped or perpetual oscillations around the steady-state[2]. This is in sharp contrast to the quasi-cyclical behavior exhibited by prices of agricultural commodities. More importantly, prices generated by cobweb models are negatively autocorrelated, though it is well known that agricultural commodities prices are positively autocorrelated (Tomek and Robinson, 2003). Due to these unrealistic price dynamics, the traditional cobweb model remained, at best, a pedagogical tool with limited applicability to real-world agricultural commodity markets.

However, the popularity of nonlinear dynamics and chaotic systems in the late 1980s revived interest in the cobweb model. This stream of literature, popularly known as chaotic cobweb-type models, introduced non-linearities into the supply or demand functions while keeping the essential features of the traditional cobweb model (i.e., production lags, lack of storage and static expectations). The price trajectories generated by chaotic cobweb models depict a rich array of quasi-periodic behavior analogous to observed prices. Consequently, chaotic dynamics are increasingly viewed as a plausible explanation for persistent price fluctuations in agricultural markets. More generally, chaotic systems are deterministic models that generate bounded yet "random-looking" trajectories exhibiting complex, non-periodic fluctuations and sensitive dependence to initial conditions. Over the past decades the study of chaotic dynamics in economics has emerged as an active area of research with several interesting applications in the fields of agriculture, finance, real-estate and industrial organization, etc. (e.g., Lundberg et al., 2015; Kubin and Gardini, 2013; Westerhoff and Wieland, 2010; and Elsadany, 2010).

The early literature on chaotic cobweb markets focused primarily on the effects of nonlinearities on the traditional cobweb model. This was no trivial matter, since nonlinearities are a necessary but not a sufficient condition for chaos. For instance, Jensen and Urban (1984) and Chiarella (1988) studied chaotic dynamics in a cobweb model with nonlinear supply and linear demand functions. Finkenstadt (1995), among others, examined the effects of nonlinear demand functions in cobweb models. Likewise, Hommes (1994) looked at chaotic cobweb models with both nonlinear supply and demand functions. Often, ad hoc mathematical complexities, e.g., an s-shaped supply curve in Chiarella (1988) or an *arctan* supply function in Hommes (1991), were introduced into the traditional cobweb model to characterize the onset of chaos, the nature of strange attractors and bifurcations. Although rigorous, no attempt was made to enrich the basic structure of the traditional cobweb model or evaluate price dynamics against observed commodity prices. The early literature gradually faded away and, by the mid 90s, the pursuit of mathematical curiosities and unusual routes to chaos was replaced by an emphasis on enriching the simplistic structure of the traditional cobweb model.

The contemporary literature on cobweb-type models is primarily geared towards capturing key features of agricultural markets or incorporating farmer's behavioral characteristics like risk aversion, bounded rationality, heterogeneity and learning into the traditional cobweb model. For example, Brock and Hommes (1997) develop a cobweb model with linear demand and supply curves in which farmers may switch between forecasts based on naïve or rational expectations.

[2]It may seem that fixed oscillations around an equilibrium state are an acceptable pattern. However, as soon as a price cycle is detectable, farmers can earn higher profits by acting counter-cyclically, thereby breaking the periodicity in prices. Convergence to a steady state, though consistent with neoclassical economics, contradicts the observed price data, which is best characterized as quasi-periodic.

The tradeoff between different expectation regimes endogenously generates nonlinearities in the traditional cobweb model, leading to chaos, aptly described by the authors as a *rational route to randomness*. Dieci and Westerhoff (2009; 2010) are most closely related to our paper. They develop a model with two interacting linear cobweb markets in which farmers with naïve expectations choose to produce either of the two goods based on profit differentials at the beginning of the production cycle. Similar to Brock and Hommes (1997), non-linearities are endogenously generated by allowing suppliers to "switch" between markets[3]. They show that interactions between horizontally linked cobweb markets contribute to the quasi-cyclicality and unpredictability observed in real world prices. In a related paper, Lundberg et al. (2015) arrive at similar conclusions in a model of interacting cobweb markets with land-use competition between food and bioenergy crops, crop switching costs and an isoelastic demand function.

Other interesting applications of chaotic cobweb models in agricultural economics include: consistency of different types of static expectation regimes (Hommes, 1998), slow adjustment towards optimal production (Onozaki et al., 2000), farmers' attitudes towards risk aversion (Boussard, 1996), naïve expectations in a competitive storage model (Mitra and Boussard 2012) and heterogeneous speculators (Westerhoff and Wieland, 2010). By explicitly capturing key aspects of the economic and institutional environment, the burgeoning literature on chaotic-cobweb models has undoubtedly increased our understanding of agricultural markets. However, despite insightful economic modeling and careful mathematical analysis, the existing literature suffers from two important shortcomings.

First, the consequences of vertical linkages in agricultural value chains, (i.e., interactions between upstream and downstream farmers) have been largely overlooked. Vertical linkages arise in an agricultural value chain when the primary input for a downstream agricultural sector is produced by another related upstream agricultural sector. Familiar examples include chick and broiler production in the poultry industry or breeding and fattening farms in the beef industry[4]. These vertical linkages are fundamentally different from the well-known vertical relationships between farmers and retailers, because retailers are not involved in agricultural production and merely serve as distributors of agricultural commodities. In the case of poultry chicken and beef sectors, agricultural production takes place at both the upstream (chick and calves) and downstream (broilers and bulls) sectors. Therefore, cobweb models are suitable candidates to analyze production decisions at different levels of vertically linked agricultural value chains. However, the effects of the mutual interdependencies between upstream and downstream farmers on price fluctuations are not straightforward because production decisions of the downstream farmers are affected by the supply of upstream products. Likewise, upstream prices are also

[3]Although insightful, in practice, asset specificity and sunk costs in agricultural production weaken horizontal supply-side linkages and switching between different commodity markets is not always possible. For example, corn cannot be cultivated on rice paddies and vice versa, despite favorable profit differentials. Similarly, although chicken and beef markets are linked from the demand side, poultry farmers cannot switch between beef and poultry production due to the specific nature of commercial production technologies.
[4]An example of vertical supply side linkages in a non-agricultural value chain is land developers and construction firms in the real estate industry.

affected by the (expected) price of downstream products. Furthermore, different lengths of downstream and upstream production cycles pose another interesting modeling challenge[5].

Second, from an empirical perspective, no concerted effort has been made to assess whether chaotic cobweb models capture the stylized features of agricultural commodity prices. Often a few passing comments on the randomness, unpredictability and quasi-cyclicality of simulated prices are deemed sufficient, without mentioning other important features of agricultural commodity prices such as positive first-order autocorrelation, relatively higher variation in upstream prices, positive skewness and negative kurtosis. The lack of an empirical context is surprising because negatively correlated prices are a major criticism of cobweb-type models. One exception is Mitra and Boussard (2012), who, in a somewhat different context, benchmark the moments of prices generated from competitive storage models with naïve expectations against observed prices, concluding that the salient characteristics of agricultural commodity prices, particularly positive autocorrelation, cannot be reproduced without storage. A cursory look at the existing literature corroborates their conclusions, as data generated from chaotic cobweb models exhibit negative or, at best, very low autocorrelation (Hommes 1998 and Brock et al. 2007). The ability of chaotic cobweb models to replicate the stylized features of agricultural commodity prices thus remains an open research question. Until this question is adequately addressed, the theory of endogenous price fluctuations will remain relegated to the periphery of mainstream price fluctuation theories.

This paper is primarily interested in the organization of agricultural production in less developed countries and addresses the abovementioned gaps in the existing literature by incorporating vertical linkages in the traditional cobweb model. Under very general assumptions, we show that a cobweb model of vertically linked markets has the potential to generate positively autocorrelated, chaotic price trajectories analogous to observed agricultural product prices. Of course, the question of vertical linkages does not arise if agricultural value chains are vertically integrated, as is often the case in developed countries. In those cases, coordination of production decisions diminishes uncertainty between upstream and downstream sectors, and a standard cobweb model with a production lag equal to the sum of the production cycles at the upstream and downstream sectors suffices. However, the standard cobweb model is not suitable for studying price fluctuations in less developed countries, where lack of vertical integration is endemic to agricultural value chains. To this end, we develop a parsimonious, dynamical model to examine the effects of mutual interdependencies between upstream and downstream farmers operating in vertically linked cobweb-type market on the associated price dynamics.

Although the model is general, the Pakistani poultry (chicken) sector is employed to provide an empirical context to the model[6]. This choice is motivated by several observations. First, the market for poultry products is primarily a live-bird market in less developed countries due to

[5]For example, the length of the production cycle of broilers is approximately twice that of chicks. Similar asymmetric production cycles are found in beef production (production of young stock and fattening of heifers) and real estate (land development and construction of houses).

[6]Parameters of agricultural production vary significantly from one industry to another. The downstream production cycle is longer than the upstream production cycle in poultry production but the opposite is true in the beef production. Analytical results on local dynamics and global price dynamics are influenced by these production benchmarks, i.e., length of production cycles, production technology, etc. Moreover, calibrations are necessitated by the fact that numerical analysis is eventually needed to study global dynamics of nonlinear dynamical systems.

lack of appropriate cold-storage facilities. Hence, poultry products at the downstream (broilers) and upstream (chicks) levels are best classified as continuously produced, non-storable commodities. Second, compared to other agricultural commodities, both upstream and downstream sectors have a relatively short production cycle, i.e., 3 weeks for chicks and 6 weeks for broilers. Naïve forecasts of future prices over such a short interval by boundedly rational farmers seem quite reasonable, given the high cost of information acquisition and processing in less developed countries. These observations clearly suggest that a cobweb-type model is a suitable candidate for modeling prices fluctuations in the poultry sector of less developed countries. Lastly, poultry chicken is the cheapest and healthiest source of animal protein, whereas, poultry production is also known to be least detrimental to the environment among livestock (Farrell 2013). Consequently, the poultry sector is economically relevant and plays an increasingly important role in less developed countries beset by rising populations and existing deficiencies in consumption of animal proteins.

We first solve for the equilibrium of the model in order to derive a system of non-linear time-delay difference equations characterizing the underlying price dynamics in vertically linked, cobweb markets. Thereafter, analytical methods and numerical tools commonly employed to nonlinear systems are used to shed light on the properties of the dynamical system. We find that vertical couplings between downstream and upstream cobweb markets and the associated time-delays have profound effects on local stability, global behavior and onset of chaos. For instance, in the special case of quadratic costs, the Hale et al. (1985) stability criterion for linear time-delay systems shows that time-delays stabilize the underling system. The phase-space and time-space plots, bifurcation diagrams and maximal Lyapunov exponent reveal complex dynamics and onset of chaotic behavior via a Hopf bifurcation, as technological efficiency and sensitivity to broiler prices cross a critical threshold. The simulations confirm that the model reproduces the patterns observed in the actual price data, i.e., quasi-cyclical behavior, positive first-order autocorrelation, relatively higher variation in upstream prices, low skewness and negative kurtosis. These findings have important ramifications vis-à-vis the literature on endogenous price fluctuations.

The remainder of the paper is organized as follows: in section 2 we develop a model of vertically linked cobweb markets, using the case of the poultry sector. The respective optimization problems of downstream and upstream farmers are solved under naïve expectations to derive the coupled laws of motion for upstream and downstream prices. Mathematical analysis and numerical simulations are used to shed light on the dynamical properties of the resulting system of time-delay difference equations in section 3 and section 4. Thereafter, we benchmark price dynamics generated by the underlying model against the stylized features of Pakistani poultry prices. We conclude the paper with a summary of our major findings along with some important caveats and suggestions for future extensions.

2. A Model of Vertically linked Cobweb Markets in Less Developed Countries

We extend the traditional cobweb model by looking at the interactions between two vertically linked agricultural markets, i.e., upstream and downstream markets. These interactions arise when an upstream product is the primary input for downstream production. For example, in the absence of vertical integration in less developed countries, chicks purchased from hatcheries are grown into broilers and sold to retailers by broiler farmers. The effects of vertical linkages on price fluctuations are not immediately obvious in cobweb-type models, because, in addition to the standard upstream and downstream cobweb effects, the supply of downstream products is affected by the prices of upstream products, while upstream prices are also affected by the (expected) price of downstream products. The different lengths of the upstream and downstream production cycles add another layer of complexity into the price dynamics. In this section we develop a parsimonious model to crystallize the effects of the aforementioned vertical linkages between upstream and downstream sectors on price dynamics.

2.1 Preliminaries

In order to focus on the effects of vertical linkages on price fluctuations, we closely follow the assumptions of the traditional cobweb model. Traditional cobweb models describe price dynamics in a market of a non-storable commodity that takes one unit of time to produce (usually normalized to the length of the production cycle), where suppliers are assumed to form expectations based on current or past prices. For sake of simplicity, we assume naïve expectations, as relaxing this would complicate analysis without adding new insights[7].In the spirit of cobweb models, the model is specified in discrete time to account for the delayed supply-response inherent to agricultural production. Quantities and prices are determined sequentially in a linked causal chain due to time lags between the decision to produce and realization of actual production plans.

We abstract from strategic interactions by assuming perfectly competitive upstream and downstream markets with many profit-maximizing farmers. Given the institutional environment of less developed countries, there is no commodity storage in our model. Consequently, current supply and demand determine market-clearing prices in both markets. We assume downstream farmers face a negatively sloped linear consumer demand curve. A linear demand function aptly captures behavior of consumers in less developed countries, wherein demand for food-based commodities is relatively elastic when prices are high (due to high poverty levels, even small changes in prices can lead to large changes in consumption) and relatively inelastic when prices are low (poor households shift income to other products once a minimum amount of food is purchased)[8]. Finally, both upstream and downstream producers face convex cost functions.

Our framework is quite general. However, in order to build better intuition, the theoretical model in this paper is based on the poultry sector. This allows us to quantify explicitly the asymmetric time delays arising from the different durations of upstream (chick) and downstream (broiler) production cycles[9]. In the poultry industry in LDCs, chick farms or hatcheries incubate

[7]A more general starting point would be a cobweb model with adaptive price expectations (Nerlove 1958). However, this would introduce additional complexity into the model, as one would need to keep track of previous lags of upstream and downstream prices, leading to an increase in the dimensionality of the system, without substantially modifying the qualitative nature of the results.

[8] For a linear demand function the elasticity of demand changes as we move across the demand curve. In particular, the elasticity is high when the prices are high, the elasticity is unitary at the midpoint of the demand curve and elasticity is low when prices are low.

[9] Asymmetric production cycles are also found in the beef industry. For example, it takes approximately 2 years to raise a flock of heifers at the upstream level. At the downstream level, heifers are kept in fattening farms and slaughtered for beef after another 6 months. Similarly, in the real-estate sector, land development at the upstream level takes longer than the actual construction of houses at the downstream level.

fertilized eggs in control sheds for exactly 3 weeks. The eggs hatch on the 22nd day and the resulting chicks are sold to broiler farms immediately, as hatcheries are not equipped to handle chicks. It takes broiler farmers another 6 weeks to grow the flock of chicks into broilers and in the absence of cold storage facilities, broilers are also sold without much delay in (live-bird) chicken markets upon reaching market weight. Following the cobweb literature, for the purposes of expositional clarity and tractability, one unit of time is normalized to represent the length of the (shorter) chick production cycle in the theoretical model, i.e., $\Delta t = 3$ weeks, such that the broilerproduction cycle is equal to two units of time.

Let us now formalize the model. Let ω_t^C and p_t^B denote upstream and downstream prices at period t, respectively. Likewise, let q_t^C and q_t^B denote output of upstream and downstream sectors at period t, respectively. It is well known that farmers' response to price signals is delayed by the fixed production cycle, often spanning several time periods, of agricultural commodities. Therefore, production is largely driven by expectations of future prices, and consequently, current output is constrained by past decisions. In fact, expectations introduce feedback effects because expectations of future prices today affect prices tomorrow, while prices tomorrow affect expectations of future prices the day after tomorrow and so on. Put simply, future prices are a function of current expectations. For instance, if τ_1 represents the length of the production cycle of upstream commodities, then the conditional expectation of upstream prices at the time of harvest, formulated at the beginning of the production cycle i.e. $E_t\left(\omega_{t+\tau_1}^C\right)$ will affect the decisions of upstream farmers in period t and hence prices in period $t+\tau_1$. As a notational convenience, we define $\widetilde{\omega}_{t+\tau_1}^C = E_t(\omega_{t+\tau_1}^C)$. Clearly, under naïve expectations: $\widetilde{\omega}_{t+\tau_1}^C = \omega_t^C$. Similarly, $\tilde{p}_{t+\tau_2}^B$ represents the conditional expectation of downstream prices in time $t+\tau_2$, formulated at time t, i.e., $E_t\left(p_{t+\tau_2}^B\right)$, and by naïve expectations $\tilde{p}_{t+\tau_2}^B = p_t^B$.

The total cost function of the representative chick farmer is denoted by $C_1(q_t^E) = (q_t^E)^\alpha$, where q_t^E representsthe number of fertilized eggs purchased by chick farmers at the beginning of the chick production cycle. Likewise, the total cost function of the representative broiler farmer is given by $C_2(q_t^C) = (q_t^C)^\beta$, where q_t^C is the number of chicks purchased at the beginning of the production cycle. In order to guarantee sufficiency of first order conditions, we assume that $\alpha > 1$ and $\beta > 1$ so that both cost functions are strictly convex. Retail demand for broilers at time t is denoted by $F(p_t) = a - bp_t^B$ such that $a > 0$ represents the maximum capacity of retailers and $b > 0$ represents the sensitivity of retail demand to broiler prices. Lastly, N_1, N_2 and N_3 denote the number of upstream farmers, downstream farmers and retailers in the industry, respectively[10].

2.2 Theoretical Model

We now solve the respective optimization problems of chick and broiler farmers in order to derive a system of difference equations characterizing price dynamics in a vertically linked cobweb market. Notice that, due to the biological nature of the broiler production, i.e., rearing of

[10]Since we are interested in short-run price dynamics only, the number of chick farmers, broiler farmers and retailers are exogenously fixed and not endogenously determined by the zero-profit (free entry) condition.

chicks into broilers, production plans formulated in time t will be realized in time $t+2$[11]. Of course the market situation may change over the duration of the production cycle, introducing *ex-ante* uncertainty into the profit maximization problem at time t.A representative broiler farmer solves the following profit maximization problem:

$$\underset{(q_{t+2}^{B})}{\text{Max}}\ \pi_{t+2}: q_{t+2}^{B}\tilde{p}_{t+2}^{B} - q_{t}^{C}\omega_{t}^{C} - (q_{t}^{C})^{\beta}$$

subject to

$$q_{t+2}^{B} = k_1 q_{t}^{C}, 0 < k_1 < 1$$

In words, the broiler farmer has to decide how many broilers to produce given the price of chicks at time t subject to a simple fixed proportions production technology that converts chicks purchased at time t into broilers at time $t+2$ with a conversion rate of k_1. As mentioned before, production costs, comprising primarily of feeding costs, are a convex function of the number of chicks purchased at time t. We substitute the broiler farmers' production function into the profit equation and use the Hotelling Lemma to derive the supply function for a representative broiler farmer[12]:

$$\frac{\partial \pi_{t+2}}{\partial q_{t+2}^{B}} = \tilde{p}_{t+2}^{B} - \frac{\beta}{k_1}\left(\frac{q_{t+2}^{B}}{k_1}\right)^{\beta-1} - \frac{\omega_{t}^{C}}{k_1} = 0 \Rightarrow q_{t+2}^{B} = k_1\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}(k_1\tilde{p}_{t+2}^{B} - \omega_{t}^{C})^{\frac{1}{\beta-1}}$$

It is straightforward to observe that the quantity of broilers produced is increasing in $\tilde{p}_{t+2}^{B}$ (i.e. expected price in period $t+2$) and decreasing in current chick prices ω_{t}^{C} (i.e. input costs). Under the assumption of homogenous broiler farmers, summing the supply curve of a representative farmer over N_2 broiler farmers gives the aggregate supply of broilers in the poultry market at time $t+2$:

$$Q_{t+2}^{B,S} = \sum q_{t+2}^{B} = N_2 k_1\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}(k_1\tilde{p}_{t+2}^{B} - \omega_{t}^{C})^{\frac{1}{\beta-1}}$$

As mentioned before, the broiler retail demand curve for N_3 homogenous retailers is given by $F(p_t) = a - bp_{t}^{B}$. Hence, aggregate demand for broilers at time $t+2$ is $Q_{t+2}^{B,D} = N_3 F(p_{t+2}^{B}) = N_3(a - bp_{t+2}^{B})$. Recall that chicken markets in less developed countries are predominantly live bird markets with limited cold storage facilities. Therefore, due to the absence of inventory, farmers sell their output at the market clearing price. Consequently, prices are determined by current demand and supply. Equating aggregate quantity demanded and supplied for broilers in time $t+2$(i.e., $Q_{t+2}^{B,D} = Q_{t+2}^{B,S}$) gives us:

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[11]In general, production at chick and broiler farms takes place in batches. At any given time, both types of farmers possess fertilized eggs and chicks, respectively, of the same age.

[12] Note that the convexity of cost function insures that the first order condition is a sufficient condition for optimality i.e.$\frac{\partial \pi_{t+3}}{\partial q_{t+3}^{B}} \geq 0$ as long as $k\tilde{p}_{t+3}^{B} \geq C_2'(\left(q_{t+3}^{B}/_{k}\right) + \omega_{t+1}^{C}$(which represents the broiler farmers' shutdown condition) and$\frac{\partial^2 \pi_{t+3}}{\partial {q_{t+3}^{B}}^2} < 0$ always.

$$N_3(a - bp_{t+2}^B) = N_2 k_1 \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} (k_1 \tilde{p}_{t+2}^B - \omega_t^C)^{\frac{1}{\beta-1}}$$

After some simplifications, we obtain the following law of motion for broiler prices:

$$p_{t+2}^B = \frac{a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 k_1}{bN_3} (k_1 \tilde{p}_{t+2}^B - \omega_t^C)^{\frac{1}{\beta-1}}$$

This shows that the conditional expectation of future broiler prices at time t (i.e., $\tilde{p}_{t+2}^B = E_t(p_{t+2}^B)$) affects broiler prices at $t+2$. Clearly, under the naïve expectations hypothesis $\tilde{p}_{t+2}^B = p_t^B$. Also note that expectations of chick prices do not enter the law of motion for broiler prices because broiler farmers do not face any uncertainty vis-à-vis chick prices because chick prices are already known at the beginning of the broiler production cycle. The law of motion for broiler prices may be recast as (since it holds for all time t):

$$p_t^B = \frac{a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 k_1}{bN_3} (k_1 p_{t-2}^B - \omega_{t-2}^C)^{\frac{1}{\beta-1}} \qquad (1)$$

The interrelationships between different prices in equation1 are intuitive. For instance, it is easy to see that $\frac{\partial p_t^B}{\partial p_{t-2}^B} < 0$, as long as $k_1 p_{t-2}^B - \omega_{t-2}^C > 0$ (clearly under naïve expectations broiler farms would shut down if expected inflow from sale of broiler is less than the cost of chicks). This represents the standard cobweb phenomenon, i.e., under naïve expectations, broiler farmers' increase planned production if current broiler prices are high, anticipating high prices to persist till the end of the production cycle. This results in a supply glut at the end of the production cycle and, as a result, broiler prices plummet. Therefore, if prices were high when production decisions were made (at time$t-2$), prices will be low at the end of the production cycle (time t) and vice versa, while $\frac{\partial p_t^B}{\partial \omega_{t-2}^C} > 0$ highlights the fact that upstream prices affect the supply of downstream products and hence downstream prices. Intuitively, all else equal, if chick prices (i.e. input costs) were high (low) when production decisions were made, then profit maximizing broiler farmers will reduce (increase) procurement of chicks, resulting in lower (higher) output of broilers at the end of the production cycle and hence higher (lower) broiler prices. Credit constraints offer another explanation for this effect. For instance, liquidity constrained broiler farmers have fixed budgets for procurement of chicks, and if chick prices are high, fewer chicks are purchased resulting in a shortage of broilers at the end of the production cycle, hence high broiler prices and vice versa.

Evidently, prices at different levels of an agricultural value chain are linked and a failure to account for these linkages can lead to erroneous conclusions. Therefore, in order to close the model and fully specify the price dynamics in the poultry value chain, we need to understand the dynamics of chick prices. The chick farmers profit maximization problem at time t can be written as:

$$\underset{(q_{t+1}^C)}{\text{Max}}\ \pi_{t+1}: q_{t+1}^C \widetilde{\omega}_{t+1}^C - (q_t^E)^\alpha - e q_t^E$$

Subject to

$$q_{t+1}^{C} = k_2 q_t^{E},\ 0 < k_2 < 1$$

In words, chick farmers decide how many chicks to produce next period or, more precisely, how many fertilized eggs to incubate at the beginning of the chick production cycle. We assume that the cost of fertilized eggs is constant at e per unit and a fixed proportion production technology converts fertilized eggs in period t into chicks in period $t+1$ at a conversion rate of k_2. The first order condition yields the optimal supply curve for the chick farmers:

$$\frac{\partial \pi_{t+1}}{\partial q_{t+1}^{C}} = \tilde{\omega}_{t+1}^{C} - \frac{\alpha}{k_2}\left(\frac{q_{t+1}^{C}}{k_2}\right)^{\alpha-1} - \frac{\mathrm{e}}{k_2} = 0 \Rightarrow q_{t+1}^{C,S} = k_2\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}(k_2\tilde{\omega}_{t+1}^{C} - e)^{\frac{1}{\alpha-1}}$$

Note that the quantity of chicks supplied is increasing in the expected price of chicks, the standard cobweb effect. Assuming chick farmers are homogenous and summing over the N_1 chick farmers we get the aggregate supply of chicks in period $t+1$.

$$Q_{t+1}^{C,S} = \sum q_{t+1}^{C,S} = N_1 k_2\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}(k_2\tilde{\omega}_{t+1}^{C} - e)^{\frac{1}{\alpha-1}}$$

Intuitively, broiler prices may also affect chick prices. Since, all else equal, higher downstream prices increase downstream farmers' willingness to pay for upstream products, the resulting higher (derived) demand for upstream products leads to higher upstream prices and vice versa. In order to formalize this intuition, we note that in a vertically linked value chain the demand for upstream products is derived from the solution to the profit maximization problem of downstream farmers (Gardner 1975). Therefore, we derive the demand for chicks from the representative broiler farmer's profit maximization problem. Substituting the broiler production function into the broiler farmers' profit equation described in the beginning of this section and applying the Shephard's Lemma yields the quantity of chicks demanded by broiler farmers:

$$\frac{\partial \pi_{t+3}}{\partial q_{t+1}^{C}} = k_1\tilde{p}_{t+3}^{B} - \omega_{t+1}^{C} - \beta(q_{t+1}^{C})^{\beta-1} = 0 \Rightarrow q_{t+1}^{C,D} = \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}(k_1\tilde{p}_{t+3}^{B} - \omega_{t+1}^{C})^{\frac{1}{\beta-1}}$$

The aggregate demand for chicks is given by: $Q_{t+1}^{C,D} = \sum q_{t+1}^{C,D} = N_2\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}(k_1\tilde{p}_{t+3}^{B} - \omega_{t+1}^{C})^{\frac{1}{\beta-1}}$. Recall that chicks are a non-storable commodity and sold to broiler farmers immediately upon hatching. Hence, by the market clearing condition, in any given period aggregate demand of chicks is equal to aggregate supply of chicks i.e. $Q_{t+1}^{C,S} = Q_{t+1}^{C,D}$:

$$N_2\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}(k_1\tilde{p}_{t+3}^{B} - \omega_{t+1}^{C})^{\frac{1}{\beta-1}} = N_1 k_2\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}(k_2\tilde{\omega}_{t+1}^{C} - e)^{\frac{1}{\alpha-1}}$$

After some algebra, the law of motion for chick prices can be expressed as:

$$\omega_{t+1}^{C} = k_1 \tilde{p}_{t+3}^{B} - \beta \left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2 \widetilde{\omega}_{t+1}^{C} - e)^{\frac{\beta-1}{\alpha-1}}$$

Note that chick prices at time $t+1$depend on the expectations of both chick and broiler prices at time t. Because, unlike broiler farmers, chick famers face uncertainty with regards to both future chick and broiler prices, which can be construed as supply-side and demand-side uncertainties, respectively. The former is the standard cobweb phenomenon, whilst the latter captures the positive demand-side effect of expected downstream prices on upstream prices. Note that by the naïve expectations hypothesis $\widetilde{\omega}_{t+1}^{C} = \omega_t^C$ and $\tilde{p}_{t+3}^{B} = p_t^B$, substituting this into the equilibrium relationship above and re-indexing the time subscripts (since it holds for all time t), we get the difference equation for chick prices:

$$\omega_t^C = k_1 p_{t-1}^{B} - \beta \left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2 \omega_{t-1}^{C} - e)^{\frac{\beta-1}{\alpha-1}} \qquad (2)$$

Intuitively, $\frac{\partial \omega_t^C}{\partial \omega_{t-1}^C} < 0$ represents the cobweb effect (since $k_2 \omega_{t-1}^C - e > 0$ by the chick farmers shutdown condition). That is, chick farmers increase production if chick prices are high at the beginning of the chick production cycle, leading to a supply glut and hence low chick prices at the end of the chick production cycle, and vice versa. All else equal, $\frac{\partial \omega_t^C}{\partial p_{t-1}^B} > 0$ stems from vertical linkages in the poultry value chain and captures the positive effect of current broiler prices on the demand for chicks in future and hence chick prices. More specifically, broiler farmers are willing to pay more for chicks in view of previously high broiler prices and vice versa under the naïve expectation that high (low) broiler prices in the past translate into high (low) broiler prices in the future. Another related interpretation is that upstream farmers are able to successfully bargain with downstream farmers for higher prices for upstream products if downstream prices are high in previous periods and vice versa.

2.3. System of (coupled) Nonlinear Time-Delay Difference Equations

Admittedly, our simplified framework overlooked several empirically relevant aspects of agricultural markets such as capacity constraints, adjustment costs, market power, risk aversion and free entry/exit etc. However, parsimony is a virtue in modeling, as it allows us to focus our attention towards the effects of vertical linkages in agricultural production on price fluctuations under static expectations. To this end, equation (1) and (2) together form a system of (coupled) nonlinear time-delay difference equations characterizing price dynamics in vertically linked cobweb-type poultry markets.

$$Price\ Dynamics \begin{cases} p_t^B = \frac{a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 k_1}{b N_3} (k_1 p_{t-2}^B - \omega_{t-2}^C)^{\frac{1}{\beta-1}} \\ \omega_t^C = k_1 p_{t-1}^{B} - \beta \left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2 \omega_{t-1}^{C} - e)^{\frac{\beta-1}{\alpha-1}} \end{cases}$$

A few important observations are as follows. First, the seemingly complicated difference equations were derived from a simple model of profit maximization by naïve upstream and downstream farmers. Second, time delays, couplings and nonlinearities in the dynamical system arise naturally from the core characteristics of agricultural markets as opposed to arbitrary assumptions made for the sake of generating "interesting" results. For instance, time-delays, driven by the unequal lengths of chick and broiler production cycles, represent the delayed-supply response inherent to agricultural production. While, couplings capture the cross-price affects stemming from the vertically linked structure of agricultural value chains[13]. And nonlinearities are generated by the convexity of cost functions[14].

From a technical aspect, in contrast to a "standard" system of difference equations, whereby, future states are completely determined by the current state, the evolution of time-delay systems also depends upon the past states of the system. Hence, techniques commonly used to examine the behavior of "standard" dynamical systems are not directly applicable to systems with time-delays. We describe some analytical methods to study the dynamical properties of time-delay systems in the next section. Also note that t does not explicitly enter as an argument in either difference equation, hence the dynamical system belongs to the category of time autonomous systems. Therefore, the underlying system can be easily rewritten in the more natural configuration with $\Delta t = 1$ week:

$$Price\ Dynamics \begin{cases} p_t^B = \frac{a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 k_1}{b N_3} (k_1 p_{t-6}^B - \omega_{t-6}^C)^{\frac{1}{\beta-1}} \\ \omega_t^C = k_1 p_{t-3}^B - \beta \left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2 \omega_{t-3}^C - e)^{\frac{\beta-1}{\alpha-1}} \end{cases}$$

It is trivial to show that both configurations are equivalent vis-à-vis dynamical properties i.e. equilibrium states, local stability and onset of chaos. However, analysis of local stability is not tractable in the case of $\Delta t = 1$ week (as will become obvious in the following pages) due to increases in the state-space brought about by the relatively longer length of the time-delays.[15] However, the latter configuration is more representative of global price dynamics in poultry markets where aggregated price data is usually available at a weekly frequency. In view of these considerations, numerical simulations in section 3.3 are based on the latter configuration.

3. Local Price Dynamics in Vertically Linked Cobweb Markets

In this section we employ mathematical analysis and numerical simulations to shed light on some important aspects of the underlying dynamical system i.e. existence of equilibrium, local stability, global dynamics and onset of chaos. Although largely overlooked in the economics

[13] Formally, two markets are said to be coupled if the supply/demand curves in one market depend on the price in the other market.

[14] It is easy to see that the system becomes linear under the special case of quadratic cost functions for both chick and broiler farmers, i.e., where$\alpha = \beta = 2$, but is non-linear in all other cases.

[15] For example, local stability analysis of the system with $\Delta t = 1$ week entails analysis of the eigenvalues of a12x12 matrix. Clearly, it is not possible to derive the characteristic polynomial in this case. We setup the corresponding 12x12 matrix in appendix 2 and numerically calculate the eigenvalues to show equivalence between the local behaviors of both configurations, i.e. ,$\Delta t = 1$ week and $\Delta t = 3$ weeks.

literature, the effects of time-delays on the behavior of dynamical systems have been studied extensively in the engineering sciences albeit in continuous-time. Time-delays are also ubiquitous in biological models of cellular automaton, epidemics and population dynamics. Campbell (2007) identified several qualitative features commonly associated with the dynamics of time-delay systems. For example, she finds that time delays often lead to delay induced oscillatory behavior created by Hopf bifurcations, existence of solutions with multiple frequencies (quasi-periodicity) and attractor switching.

Interestingly, many of the abovementioned features are also found in chaotic systems. In fact, time-delay differential equations (continuous-time analogue of time-delay difference equations) belong to the class of functional differential equations that are known to be infinite-dimensional problems characterized by complicated behavior, often without closed-form analytical solutions[16]. Although, time-delay difference equations are finite-dimensional, nevertheless, increases in the dimension of the model's underlying state-space brought about by the presence of time-delays (see section 3.2) often make it analytically intractable. And it is well known that complex dynamics are associated with high dimensional state-spaces.

3.1. Analysis of Equilibrium

We begin this section by examining the equilibrium states of the underlying system of time-delay difference equations. Note that in the steady state $p_t^B = p_{t-\tau}^B = p^B$ and similarly $\omega_t^C = \omega_{t-\tau}^C = \omega^c$ for $\tau = 1,2, \dots \infty$. Substituting this into the dynamical system and after some tedious algebra, details of which are relegated to appendix 1, we prove that the underlying dynamical system has a unique, positively valued equilibrium state.

> **Proposition 1: The underlying dynamical system possesses a unique, positively-valued equilibrium state such that $\boldsymbol{p^B > \omega^c > 0}$.**

A few important remarks relevant to the equilibrium state are as follows. First, as is obvious from calculations in appendix 1, there is no closed-form expression for the steady state values of chick and broiler prices[17]. Second, given the definition of the parameters in our model, the equilibrium state is economically relevant as both the equilibrium chick and broiler prices are non-zero and strictly positive. Likewise, as expected, in equilibrium broiler prices are strictly greater than chick prices, i.e.,$p^B > \omega^c > 0$. For sake of brevity, we do not delve into comparative static analysis.

3.2. Analysis of Local Stability

The next step is to examine the (asymptotic) local dynamics in order to determine how the system responds to small perturbations from the equilibrium state. Standard tools used in the analysis of local stability of non-linear systems are no longer applicable to time-delay systems in

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[16] The characteristic equation of time-delay differential equations is a quasi-polynomial with infinite number of roots in the complex plane. The Lypanov-Krasovski functionals, Razumikhin techniques and Pade' approximations are some commonly used analytical tools to study the stability of systems with a functional state variable, each with its own pros and cons (Zavaeri and Jamshidi 1987).

[17] The equation describing the steady state of the model is well behaved. Hence, standard numerical algorithms can be employed to compute the equilibrium values of chick and broiler prices under reasonable calibrations.

lieu of delayed feedback mechanisms. For instance, the workhorse of stability analysis, the eigenvalues of the underlying Jacobian matrix, cannot be directly applied to study time-delay systems due to the dependence on past states. Nevertheless, mathematicians have developed methods to study local dynamics of nonlinear time-delay systems. One well-known method, state-space augmentation, converts a system of time-delay difference equations into a higher-order system without delays (Emilia 2014)[18]. The eigenvalues of the resulting higher-order system, comprising of a sequence of delay-free difference equations, characterize the (asymptotic) local stability of the original time-delay system. More specifically, the original time-delay system is locally stable, if and only if, all the eigenvalues of the augmented systems lie within the unit circle (in the real-imaginary axis space) and vice-versa. See Yassen and Agiza (2003), Hassan (2004) and Elsadany (2010) for applications of this principle. However, appeal to this simplicity often comes at the cost of analytical difficulties and, as a result, numerical methods are usually needed to study the local behavior of time-delay systems, especially if the length of the delay is not small.

In appendix 2, we use a first order approximation to linearize the non-linear, time-delay system around unique equilibrium state. Thereafter, new state variables, representing time-delays in each time-delay difference equations, are defined to convert the linearized time-delay system into a higher-order system of first-order difference equations. Finally, after some mathematical computations and tedious algebra, the characteristic polynomial for the augmented system is derived as (where A, and D are constants defined in appendix 2):

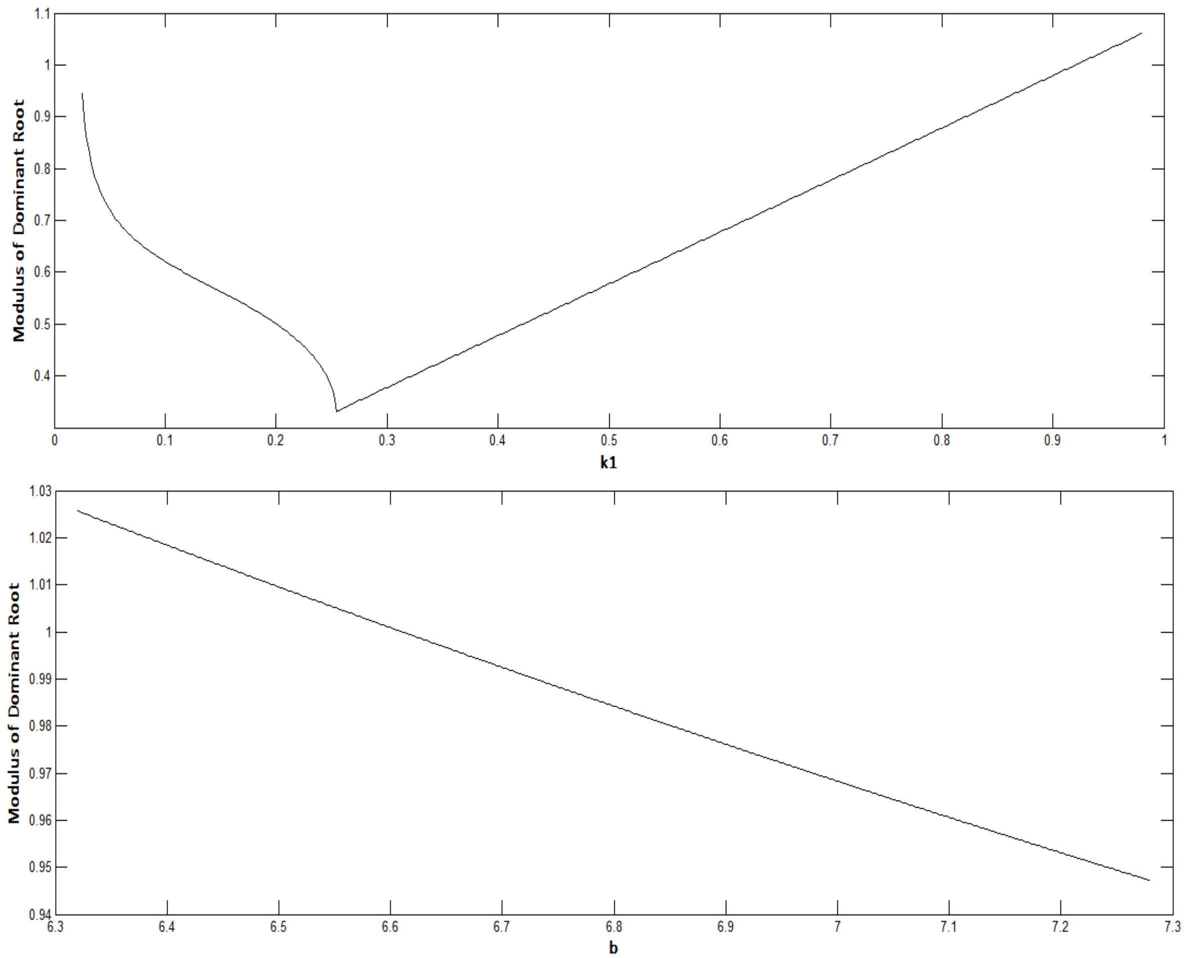
$$\lambda(\lambda^3 - D\lambda^2 - A\lambda + A(D+1))$$

The qualitative behavior of the underlying time-delay system close to the equilibrium state is determined by the roots (i.e., eigenvalues) of this characteristic polynomial. Clearly, $\lambda = 0$ is one root and this implies that either all remaining roots are also real numbers or there is one pair of complex conjugates and one real root. However, since there is no closed-form analytical expression for the fixed points of the system (p^B, ω^c), it is not possible to factor out the roots of the characteristic polynomial in terms of model parameters. Nevertheless, dependence of local (asymptotic) stability on key parameters can be studied by the means of numerical experiments. Since local asymptotic behavior is determined by the dominant root, we compute the absolute value of the largest root of the characteristic equation under reasonable calibrations for different value of k_1 (the efficiency of the broiler farmers' production technology) and b (the sensitivity of retail demand to changes in broiler prices)[19].

[18]The state-space augmentation is not applicable to continuous-time time-delay systems for obvious reasons, an inherently infinite dimensional state-space. Even in discrete time-delay systems, the state-space augmentation method runs into difficulties if delays are uncertain or variable.

[19] From an agricultural policy perspective, $k_1$and k_2 (representing technological efficiency at the upstream and downstream levels respectively) along with b (representing sensitivity of consumers to downstream prices) are the only parameters of interest in this model. All other parameters are essentially scale variables (N_1,N_2, N_3, $\alpha$and$e$) without much intuitive significance. However, the effects of changes in k_2 on system dynamics are limited and not interesting.

Figure 1: Modulus of Dominant Root with Respect to $k_1$and$b$



Top panel shows how the modulus of the dominant eigenvalue (y-axis) changes with k_1 (x-axis) and the bottom panel shows how the modulus of the dominant eigenvalue (y-axis) changes with b (x-axis) under identical model calibrations given by $\alpha = 1.75, \beta = 1.65, k_2 = 0.95, \alpha = 1000$, $e = 3, N_1 = N_2 = N_3 = 100$. Where,$b = 6.5$ in upper panel and $k_1 = 0.93$ is lower panel. We used Broyden's Method (CompEcon Toolbox by Miranda and Fackler 2003) to numerically solve for the fixed points of the nonlinear system and MATLAB roots function to compute the roots of the characteristic polynomial in these simulations.

Although figure1 represents a set of numerical experiments, it nevertheless reveals important information about the qualitative behavior of system near the steady-state. First, figure 1 shows that a wide range of behavior is possible, including (local) convergence to equilibrium for small values of k_1 (modulus of dominant roots is less than 1) and (local) diversion from equilibrium for larger values of k_1(modulus of dominant roots is greater than 1) under the given model calibrations. Conversely in the lower panel of figure 1, the system (locally) diverges from the equilibrium for small values of b and (locally) converges towards the equilibrium for larger values of b.

Second, it is important to point out that eigenvalues of the system, under both set of numerical experiments, contain a pair of complex conjugates for a large range of parameter values, i.e., the entire domain of k_1 greater than 0.25 and all values of b in the parameter range shown above. In multi-dimensional systems, complex eigenvalues are indicative of oscillatory behavior around the equilibrium state and onset of chaos via the Niemark-Sacker bifurcation occurs when at least one pair of complex conjugates cross the boundary of the unit circle (Hommes 2013). However, the system is said to be locally stable and hyperbolic if all eigenvalues are inside the unit circle. We explore issues related to chaotic dynamics more closely in the next section.

3.3 Analysis of Global Stability under the case of Quadratic Costs ($\alpha = \beta = 2$)

In linear systems the notion of (asymptotic) local and global stability are equivalent. Therefore, it is instructive to examine the effects of time-delays on global dynamics in the presence of quadratic costs, whereby the underlying dynamical system becomes linear (See Appendix 3). In particular, we want to determine whether time-delays have stabilizing or destabilizing effects on price dynamics. In the case of linear time-delays systems, Hale et al. (1985) have provided necessary and sufficient conditions for the stability independent of delays, i.e., a stability criterion for an otherwise identical system but without time-delays in feedback mechanisms. In appendix 3, we use the Hale et al. (1985) criterion to prove that the underlying (linear) time-delay model is unstable in the absence of time delays over the entire domain of parameter values. Note that this is in contrast to the equivalent linear time-delay system, which is asymptotically stable in a large subset of the domain of parameter values.

> **Proposition 2: In the absence of time-delays, the underlying linear time-delay system is asymptotically unstable independent of time-delays for all parameter values because the absolute value of the dominant eigenvalue is greater than 1.**

Put simply, Proposition 2 states that delays in feedback mechanisms stabilize an otherwise unstable price dynamic. This finding is supported by the industrial organization literature on the effects of time-delays in transmission of information related to competitors' output on the stability of the Nash equilibrium in duopoly models. For example, Yassen and Agiza (2003), Hassan (2004) and Elsadany (2010) prove that time-delays in the transmission of information on competitor's output increase the range of parameter values over which the Nash equilibrium is stable.

The primary objective of this section was to utilize analytical tools to study the dynamical properties of the underlying system of time-delay difference equations. We proved that the dynamical system possess a unique, economically relevant steady-state. State-space augmentation techniques were used to show that the dynamical system depicts a wide range of local behavior, including convergence, divergence and oscillations around the equilibrium state. Moreover, complex eigenvalues with unit modulus for a large range of parameter values point towards chaotic dynamics. Finally, in the special case of quadratic costs, time-delays have a stabilizing effect on the dynamics of the model.

4. Global Price Dynamics in Vertically Linked Cobweb Markets

In non-linear systems, "global" dynamics refer to long-term phenomena that cannot be detected by inspecting the behavior of a system in the neighborhood of the steady state. Neither are analytical methods particularly useful, especially in the case of high-dimensional systems (Puu 2013). Therefore, we turn our attention towards some well-known numerical methods (i.e. line-plots, phase diagrams, bifurcations diagrams and maximal Lyapunov exponent) to study the global behavior of the underlying non-linear, time-delay system. Model calibrations used in figure 1, referred to from here on as the baseline scenario, are employed throughout this section to facilitate comparison between the results of different numerical tools.

In the absence of reliable empirical estimates, the model was calibrated so as to maintain prices within an acceptable range, whilst ensuring that parameter values are plausible from an agricultural economics perspective. In practice, reasonable estimates of production technologies (k_1 and k_2) and cost functions (α and β) can be backed out from industry reports. Qualitative evidence from farmer surveys in less developed countries (LDCs) suggest that industry averages of k_1 and k_2 are roughly equal and high. Whereas, chick farmer's cost curves are relatively steeper compared to broiler farmer's cost curves because chick production is more capital intensive compared to broiler production (Chaudhry et al. 2016). Consistent with the assumption of perfectly competitive markets, one finds a large number of chick farmers, broiler farmers and retailers (N_1, N_2 and $N_{3,}$ respectively) in the poultry markets of LDCs. However, it is not easy to estimate parameters (a and b) of the hypothesized demand function in the absence of consumption data for LDCs. An attempt has been made to ensure that the aforementioned stylized features are reflected in the hypothetical poultry markets examined in this section.

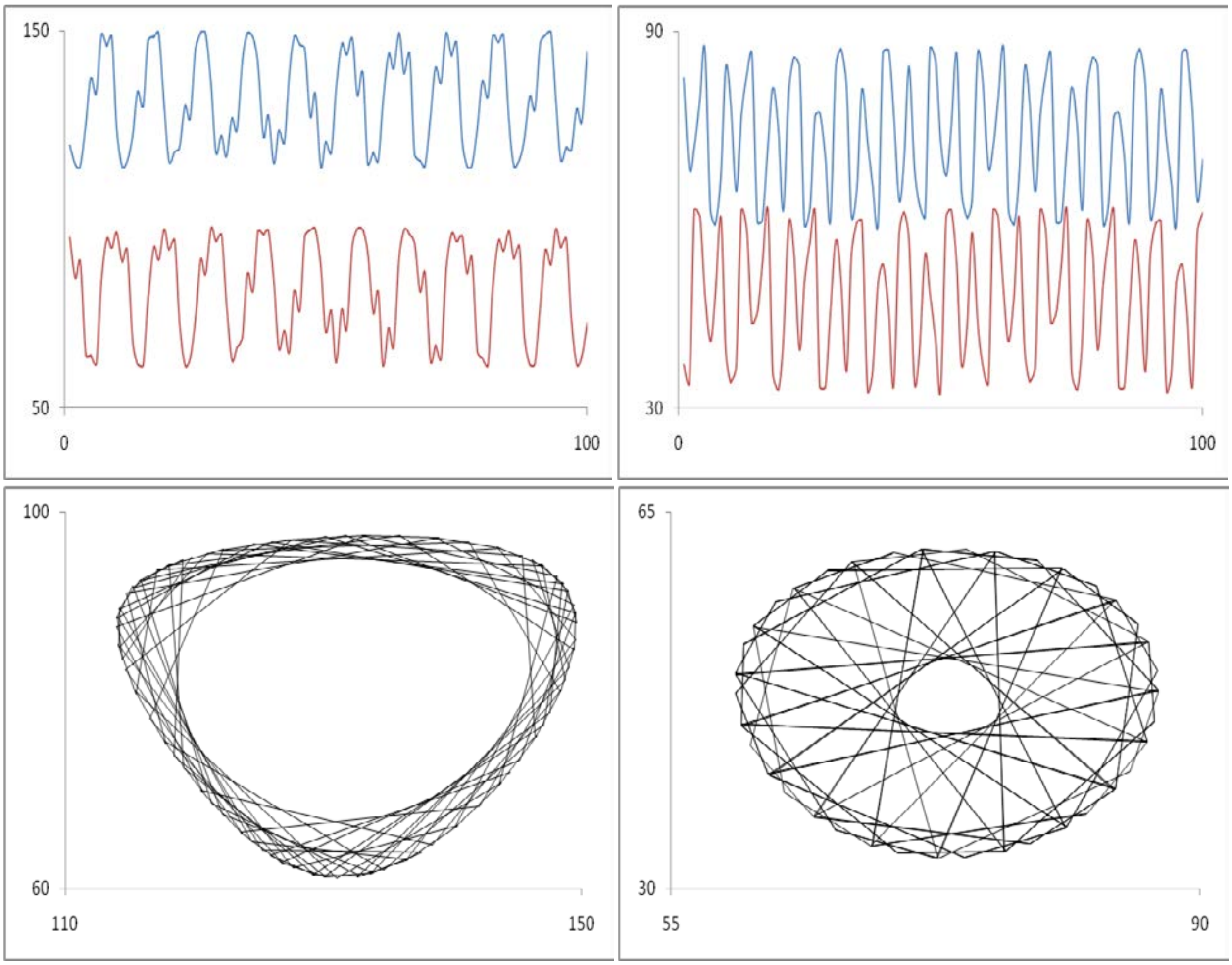
In contrast to the complicated numerical algorithms needed to solve time-delay differential equations, the orbits of a system of time-delay difference equations can be easily traced by a simple forward-looking loop. All numerical simulations are based on the configuration of $\Delta t = 1$ week, hence, a vector comprising of six initial values of chick and broiler prices (each) is needed to initialize the forward-looking loops. For sake of comparability, the vector of initial conditions is identical for all numerical simulations in this section. Initial conditions are sampled from a reasonable range within the domain of chick and broiler prices. Figure 2 depicts the dynamics of chick (red line) and broiler (blue line) prices in the underlying model in both time-domain (top panel) and phase-space (bottom panel) given two different set of model calibrations (baseline scenario on the left).

4.1 Global Dynamics in Phase-Space and Time-Space

Line-plots show that simulated prices trace bounded, non-periodic, oscillatory orbits over time. Simulated prices behave cyclically, i.e., periods of increasing prices are followed by periods of decreasing prices and vice versa. However, price cycles do not possess a fixed period or amplitude. This kind of qausi-periodic behavior is a hallmark of chaotic cobweb models. The line-plot based on the base-case scenario (top-left panel) may look periodic to the naked eye, but on closer look it is clearly not. The corresponding phase-diagram in the bottom-left panel clearly shows that values of simulated prices over time, albeit close, are never equal. Note that the simulated price trajectories do not possess a limit cycle in the phase-space, another distinguishing feature of chaotic systems. In fact, orbits encircle an apparently "*strange*" attractor in a complicated manner, without ever converging to a limit cycle or the model's unique steady state.

Very briefly, s*trange* attractors are defined as a finite subset of state-space to which trajectories of chaotic systems are pulled into overtime. A remarkable feature of chaotic systems is that even after running a large number of iterations; trajectories never visit the same point inside the strange-attractor twice, giving rise to dense-orbits in phase-space. Trajectories move within the strange attractor in a continuously unpredictable manner, such that points on the trajectories that are initially close do not remain close but diverge, come back close again but only to diverge again, and this cycle continues indefinitely. This type of dynamic equilibrium is called a *strange attractor* (Celso et al., 1987). As obvious from the phase-diagrams in figure 2, the geometric shape of strange attractors, also known as fractals, depicts both regularity and incongruity at the same time[20]. Also note that different model calibrations generate completely different types of fractals.

Figure 2: Chick and Broiler Price Dynamics in a Non-Linear Time Delay Model



Top panel shows line plots for ω_t^C (red lines) and p_t^B (blue lines),while, bottom panel shows the corresponding phase-space plots i.e. plot of ω_t^C (y-axis) against p_t^B (x-axis)under two different set of model calibrations. From left to right, parameter values for both set of calibrations are: (1) $\alpha = 1.75, \beta = 1.65, k_1 = 0.93, k_2 = 0.95, a = 1000, b = 6.5, e = 3, N_1 = N_2 = N_3 = 100$ and(2) $\alpha = 2.15, \beta = 1.95, k_1 = 0.939, k_2 = 0.939, a = 70, b = 0.795, e = 3, N_1 = N_2 = N_3 = 100$.Each set of plots is based on a vector of 100 simulated prices obtained after dropping the previous 2500 iterations to purge any transients. Axes have been appropriately scaled to highlight price dynamics.

The price dynamics depicted in figure 2 are intended for illustrative purposes only. The underlying model has the potential to produce an array of dynamics for different parameters, including convergence to the unique steady state and other types of complex dynamics.

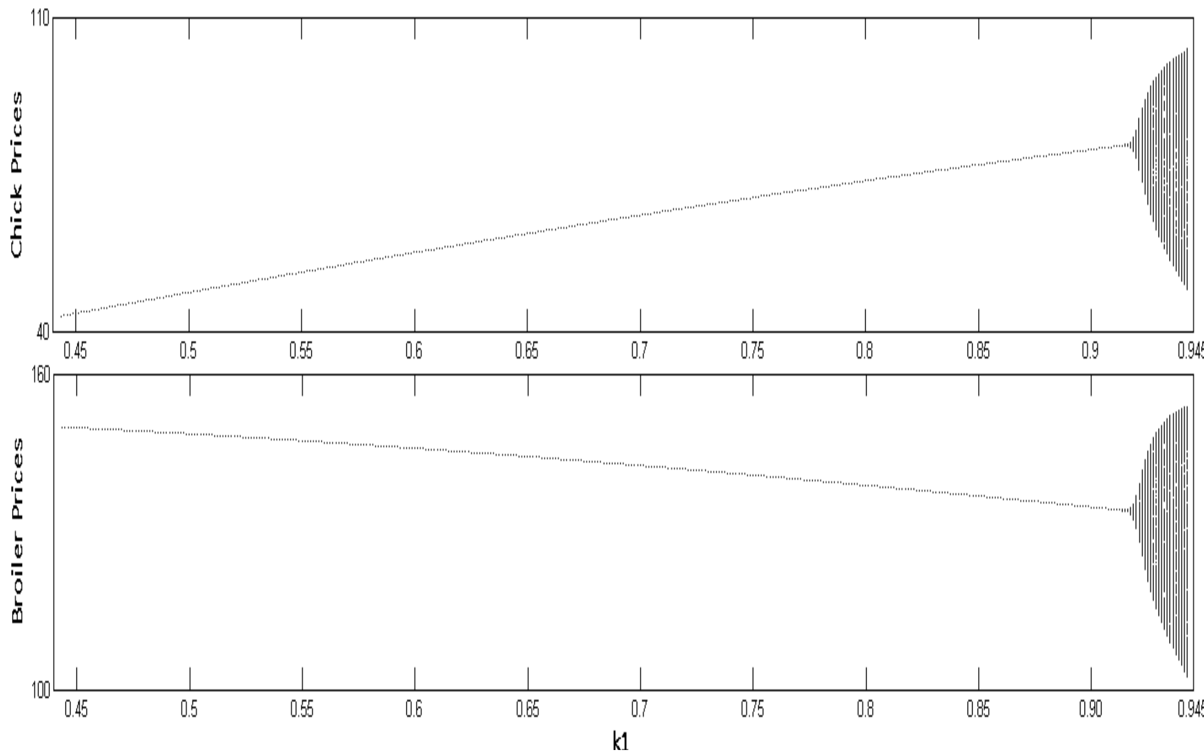
[20] Fractals are infinitely detailed geometric shapes displaying self-similarity at different levels of complexity. Unlike standard shapes, the dimension of fractals is usually a rational number. This makes their study both fascinating and cumbersome. Examining the fractal dimensions of the *strange* attractors in figure 2 is beyond the scope of this paper.

Obviously, it is not feasible to show all the different cases here. Nevertheless, in summary, visual analysis of global price dynamics in both time-space and state-space is clearly indicative of chaos given that bounded, non-periodic and dense orbits are all well-known signs of chaotic dynamics (Brock 1986). We now turn to bifurcation diagrams to gain further insights about the onset of chaos.

4.2Bifurcation Diagrams

Bifurcation diagrams summarize the limiting behavior of nonlinear dynamical systems by highlighting how the global stability of fixed points changes as a function of some parameter of interest. A bifurcation is said to have occurred if the phase-portrait of the underlying system (qualitatively) changes as a result of variations in the given parameter of interest. The bifurcation diagram of the underlying dynamical system with respect to k_1 is given in figure 3.

Figure 3: Bifurcation Diagram with Respect to k_1

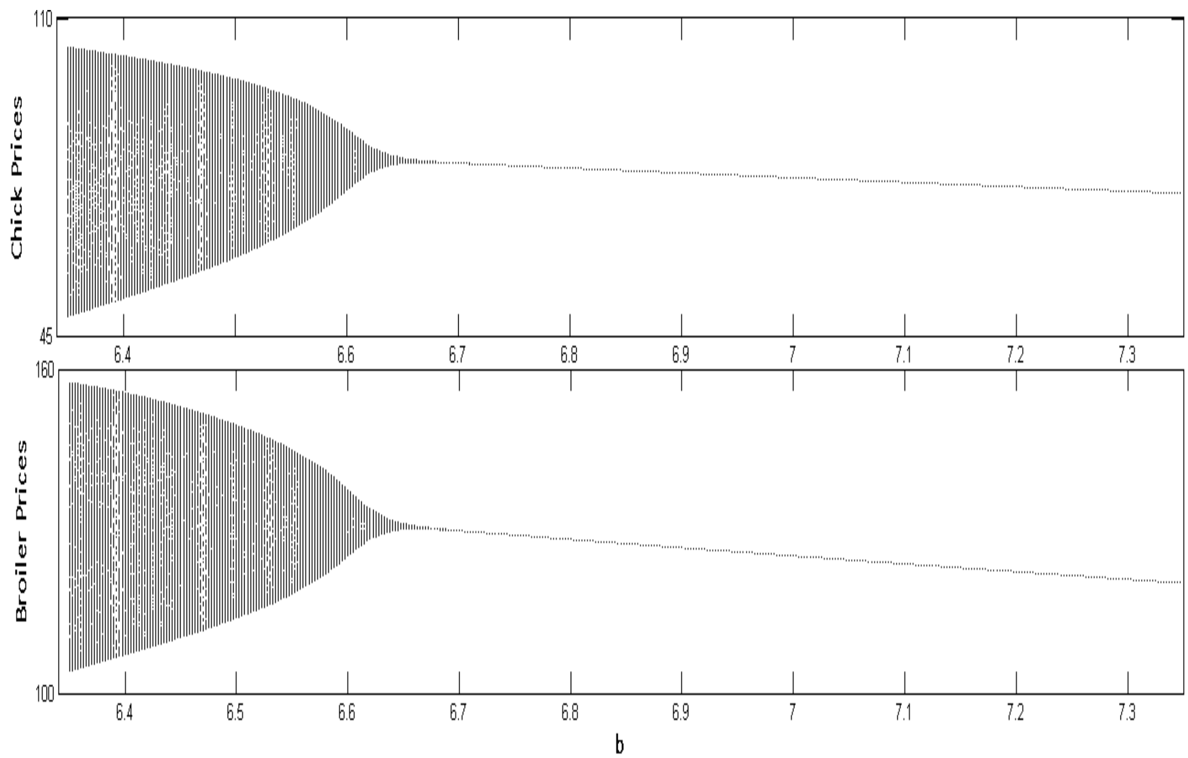


Model calibrations used to generate the bifurcation diagram are identical to the baseline scenario, except k_1. The parameter of interest i.e. k_1, was increased from 0.44 to 0.94 in 400 discrete steps. For each value of k_1, 500 observations are plotted after getting rid of any transients i.e. previous 2500 iterations were truncated to eliminate effects of any potential transients.

Figure 3 shows that the fixed point of the underlying system loses stability and quasi-periodic behavior sets in as k_1 exceeds a critical threshold. This type of behavior is indicative of a super-critical Neimark-Sacker bifurcation. Neimark-Sacker bifurcations occur in multidimensional dynamical systems when the set of eigenvalues contains a (dominant) pair of complex conjugates with unit modulus. More precisely, at the point of bifurcation ($k_1 = 0.9206$in this case), the fixed point loses stability as the (dominant) complex conjugates cross the boundary of the unit circle, typically giving birth to a locally attracting curve with quasi-periodic (or periodic) dynamics around the fixed point (Hommes 2013). Results on local stability in section 3.2

corroborate this conclusion i.e. at $k_1 = 0.9206$ the dominant root of the characteristic polynomial (under identical calibrations) is complex and has an absolute value of one (refer to top-panel of figure 1). Likewise, as predicted by the bifurcation diagram below, the phase-diagram for the baseline scenario (see top-panel of figure 2) with $k_1 = 0.93$ depicts quasi-periodic, chaotic dynamics.

Figure 4: Bifurcation Diagram with Respect to b



Model calibrations and initial conditions used to generate the bifurcation diagram are identical to the baseline scenario except b. The parameter of interest i.e. b, was increased from 6.3 to 7.3 in 400 discrete steps. For each value of b, 500 observations are plotted after erasing a much longer transient (previous 2500 iterations truncated to eliminate effects of any potential transients).

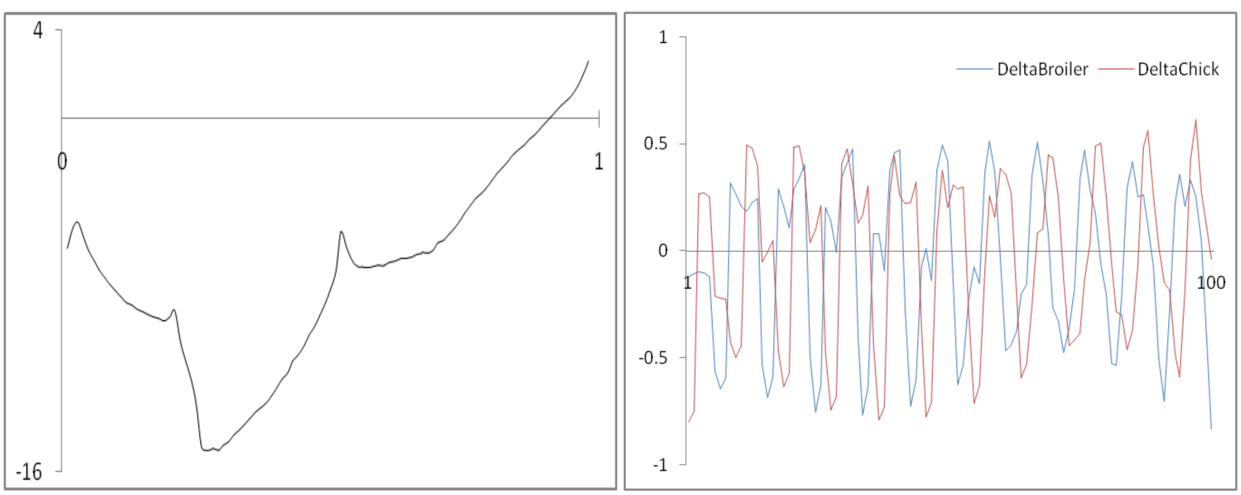
Figure 4 shows the bifurcation diagram of the underlying dynamical system with respect to b (under the baseline scenario). The transition from quasi-periodic behavior to a globally stable steady state points towards a sub-critical Niemark-Sacker bifurcation. Whereby, the fixed point is globally stable for values of b greater than 6.61 (the bifurcation point), otherwise, onset of quasi-periodic cycles destabilizes the steady state. Again, pervious results lend support to these conclusions. For instance, figure 1 highlights that the dominant eigenvalue is complex for values of b between 6.3 and 7.3, moreover, the modulus of the dominant eigenvalue decreases as b increases and equals one precisely at $b = 6.61$. Similarly, visual inspection of phase-diagrams in figure 2 (under the baseline scenario) reveal chaotic dynamics when $b = 6.5$, as suggested by the bifurcation diagram below.

The bifurcation diagrams highlight several important aspects of price dynamics generated by the underlying nonlinear, time-delay system. First, it is easy to note that chick and broiler prices are coupled and follow (qualitatively) indistinguishable dynamics. Therefore, from a policy perspective, policymakers need to account for the effects of interventions in a given market on the corresponding upstream or downstream market. Second, as mentioned before, a rich array of global behavior is possible including convergence to the unique steady state, periodic oscillations around the steady state and chaotic fluctuations. Third, chaotic dynamics do not just arise at isolated points, in which case they are difficult to observe in reality, but occur over an empirically relevant subset of the parameter space. Lastly, from an agricultural economics perspective, bifurcations with respect to k_1 (figure 3) show that the emergence of chaotic price fluctuations is linked to improvements in technological efficiency at the downstream level. Whereas, bifurcations with respect to b (figure 4) imply that chaotic dynamics are more likely to arise if consumers are less sensitive (i.e. smaller values of b) to changes in the prices of downstream agricultural products. Taken together these findings suggest that the declining sensitivity to agricultural commodity prices due to rising incomes in the modern era and rapid advances in agricultural production technology maybe behind the increasingly unpredictable behavior of agricultural commodity prices.

4.3 Dependence on Initial Conditions

Sensitive dependence on initial conditions is another defining feature of chaotic systems, whereby, small differences in initial conditions have a lasting impact on underlying dynamics. The maximal Lyapunov exponent is used to test for sensitive dependence on initial conditions. It measures the (average) exponential rate of divergence (or convergence) between two trajectories with arbitrarily close initial conditions. Positive values for the maximal Lyapunov exponent are indicative of observable chaos and negative values imply convergence.

Figure 5: Sensitive Dependence on Initial Conditions and Maximal Lyapunov Exponent



Left-panel shows the numerical approximation of the maximal Lyapunov exponent with respect to k_1 under the baseline scenario but with two arbitrarily close sets of initial conditions. Initial condition vector for the baseline model is given by (p_t^B, ω_t^C): $\{(100,20), (85,25), (79,30), (70,30), (72,27), (80,20)\}$. The other initial condition vector is given by (p_t^B, ω_t^C): $\{(102,22), (87,27), (81,32), (72,32), (74,29), (82,22)\}$. Both sets of initial conditions were randomly selected from the domain of broiler and chick prices under different values of k_1.Calculations of the maximal Lyapunov exponent are based on 100 iterations for each value of k_1, wherek_1 was increased in 100 discrete steps from 0 to 1. Left-panel shows the difference between trajectories generated by the underlying model under the baseline scenario but with abovementioned initial condition vectors.

Left-panel of figure 5 shows numerical approximations of the maximal Lyapunov exponent with respect to k_1[21]. Difference between trajectories of chick prices and broiler prices over time under the baseline scenario but with different initial conditions are depicted in the left-panel of figure 5. Visual inspection of the maximal Lyapunov exponent diagram clearly shows that onset of chaos takes place as k_1 approaches 1. This conclusion is corroborated by findings from local stability analysis (figure 1) and bifurcation diagrams (figure 3). Likewise, the right-panel confirms sensitive dependence on initial conditions under the baseline scenario ($k_1 = 0.93$) i.e. distance between trajectories under identical calibrations but different, albeit close, initial conditions does not tend to zero over time or explode.

Chaos poses major problems to the discipline of statistical modeling since it is not possible to identify precisely the "true" initial conditions and even small measurement errors in initial conditions can lead to very different price trajectories. Unfortunately, the existence of chaos in high dimensional systems cannot be established without recourse to numerical tools like phase diagrams, bifurcation diagrams and maximal Lyapunov exponent. Collectively, evidence from the aforementioned numerical tools clearly shows that complex and chaotic dynamics are not a special case, but arise over a reasonably large subset of the parameter space in the underlying model. More specifically, onset of chaos, via a Hopf bifurcation, takes place as the technological efficiency and sensitivity to downstream prices cross a critical threshold. Divergence of nearby trajectories from one another (without exploding) provides further indication of chaotic dynamics.

5. Empirical Relevance

The literature on chaotic models is often devoid of empirical content (Brock, 1986). To circumvent this criticism, we benchmark the statistical properties of prices generated by the underlying model against price data from the Pakistan poultry sector. Pakistan is a low-income, agrarian economy. The $5 billion poultry industry provides employment to approximately 1.5 million people and contributes 1.3% towards the GDP of Pakistan (Government of Pakistan 2014). The poultry market in Pakistan is primarily a live-bird market and approximately 98% of the demand for chicken is met by freshly slaughtered broilers (Haq 2014). Chick and broiler markets are competitive, and in the absence of cold storage facilities, vertically integration or pre-contracted production, current supply and demand situation determines market-clearing prices (Chaudhry et al., 2016).

The line-plot above shows weekly, farm-gate chick and broiler prices (in Pakistan rupees) for a period of 100 weeks spanning from July-2009 to May-2011.Observed prices exhibit repetitive patterns, i.e., periods of rising prices followed by periods of declining prices and vice versa. These oscillatory fluctuations are largely consistent with quasi-periodic behavior of simulated prices (see figure 2).

[21] As mentioned before, the Jacobian of time delay-systems cannot be defined analytically. Therefore, we employ numerical methods to track the exponential rate of separation between two nearby trajectories (normalized to the initial level of separation each period) over time.

Figure 6: Weekly Chick and Broiler Prices from Pakistan Poultry Industry.

Statistical properties of observed and simulated prices are summarized in Table 1. Table 1 confirms that the underlying model reproduces the stylized features of observed poultry prices. First, the autocorrelation coefficient of simulated prices is positive and high in the baseline scenario (0.60). Although, less than the autocorrelation coefficient of observed prices (0.90), nevertheless, it is a significant improvement over negatively autocorrelated prices found in the existing literature in chaotic cobweb markets. Moreover, in contrast to storage in Mitra and Boussard (2012), positively correlated prices in our model stem from vertical linkages and the associated time-delays. To put these results into perspective, note that in their seminal contribution on competitive storage models, Deaton and Laroque (1992) obtained autocorrelation coefficients of approximately 0.5. Subsequent refinements by Cafiero and Wright (2006) resulted in autocorrelation coefficient of approximately 0.7 albeit with artificially low storage costs.

Table 1: Comparison of Statistical Properties of Observed and Simulated Prices

Scenario	First-Order Autocorrelation	Coefficient of Variation	Skewness	Kurtosis
Observed broiler prices in Pakistan	0.85	0.13	0.50	-0.56
Observed chick prices in Pakistan	0.96	0.37	-0.48	-0.72
Simulated broiler prices (baseline)	0.60	0.10	0.06	-1.48
Simulated chick prices (baseline)	0.59	0.16	-0.30	-1.47
Simulated broiler prices (non-baseline)	0.14	0.14	-0.04	-1.50
Simulated chick prices (non-baseline)	0.14	0.22	-0.07	-1.52

For sake of comparison, all statistical measures are based on a set of 100 observations of observed and simulated prices. Observed farm-gate prices, obtained from Pakistan Poultry Association, represent weekly market-clearing prices of chicks and broilers during a period of 100 weeks spanning from 2 July 2009 to 26 May 2011. Statistical measures of simulated prices are based on simulated data used in figure 2.

Second, it is well known that prices of upstream agricultural products are relatively more volatile compared to prices of downstream agricultural products. Notice that the coefficient of variation of chick (upstream) prices is almost twice that of broiler (downstream) prices in both the observed and simulated data. Third, negative kurtosis, i.e., heavy tails, is another common feature of agricultural commodity prices (Mitra and Boussard 2012). The kurtosis of observed chick and broiler prices is -0.56 and -0.72, respectively, and approximately -1.50 in the simulated data. Fourth, under the baseline scenario, skewness of observed (-0.48) and simulated (-0.30) chick prices are similar. Both simulated (0.06) and observed broiler prices (0.50) are positively skewed, although the skewness of simulated broiler prices is much lower.

5.1 Model Appraisal: Limitations and Extensions

Briefly, a simple model of naïve, profit-maximizing upstream and downstream farmers, operating in a vertically linked agricultural market, has the potential to generate prices that are qualitatively similar to observed prices. Admittedly, the underlying model fails to replicate the exact data-generating process, but neither are economic models expected to do so. The primary objective of an economic model is to shed light on the *fundamental* mechanisms behind a given phenomenon of interest in a specific economic environment. Obviously, this process entails overlooking potentially important factors like adjustment costs, heterogeneous expectations, capacity constraints, market power and risk aversion.

The exercise of benchmarking simulated prices against observed prices highlights some important limitations of the underlying model. First, visual inspection of observed prices in figure 6 and simulated prices in figure 2 reveals that the “speed” at which simulated prices rise and fall is significantly greater than that of observed prices. For example, over the duration of 100 weeks, trajectories of observed prices completed approximately five cycles while, trajectories of simulated prices completed ten cycles in the same amount of time. Intuitively, adjustment costs are one plausible explanation for this discrepancy between observed prices and simulated prices. In our model, naïve farmers were assumed to perfectly adjust production in response to new price signals. However, in reality, changes in the level of production come with adjustment costs stemming from capacity issues, credit constraints and the size of the breeding herd (especially at the upstream level). Consequently, often only partial adjustment towards the optimal production levels is possible in response to a price change. The resulting “inertia” in observed prices explains the higher correlation and the fewer number of price cycles over a given period of time. This so-called “inertia” may also be driven by gradient dynamics (Elsedany 2010), whereby boundedly rational farmers partially increase production in the direction of marginal profits from previous periods. Heterogeneous expectations are another possible explanation for the “inertia” in observed prices[22].

[22] Chavas (1999a) and Chavas (2000) find strong empirical evidence in support of heterogeneous expectation regimes in US broiler and beef sectors, respectively. If farmers follow different expectation regimes (rational, adaptive or naïve), then changes in production in response to a given price change are not perfectly correlated. This implies a smaller change in production at the aggregate level for a given price change and hence a smaller change in observed prices.

Second, as a simplification the number of farmers/retailers was taken as exogenously given in the underlying model, independent of the free-entry condition, whereby farmers/retailers enter or exit the industry over time to preserve the (long-run) zero economic profit condition. Consequently, vertical shifts in the observed price levels, consistent with long-run dynamics, are largely missing from simulated prices. Finally, incorporating exogenous noise (representing shocks to the agricultural sector e.g. disease, drought etc.) into the underlying model can enrich the range of possible dynamics and give rise to more realistic patterns. For sake of brevity we choose not to pursue the aforementioned extensions here. The existing literature has examined some of these issues, although under very different settings, e.g. production adjustment costs (Onozaki et al. 2000), gradient dynamics (Elsadany 2010), farmer entry and exit (Dieci and Westerhoff2010), heterogeneous expectations (Brock and Hommes 1997) and exogenous shocks in chaotic systems (Hommes and Rosser 2001). However, it remains to be seen whether all of facets of agricultural markets can be tractably integrated into a single model.

6. Conclusion

Notwithstanding the model's weaknesses, the objective of this paper was to examine the effects of vertical linkages and asymmetric production cycles on the dynamics of agricultural commodity prices, an aspect of agricultural value chains that has been largely ignored by the theoretical literature on chaotic cobweb markets. To this end, we developed a parsimonious, dynamic model to capture mutual interdependencies between upstream and downstream farmers arising from the vertically linked structure of agricultural values chains. The model was solved under the naïve expectations hypothesis to derive a system of coupled, time-delay difference equations characterizing price dynamics in vertically linked cobweb markets. Thereafter, mathematical analysis and numerical tools were employed to study the dynamical properties of the underlying system. We proved that the underlying system has a unique steady state and state-space augmentation methods revealed that the system is locally (asymptotically) stable for a large range of parameter values. Moreover, in the special case of quadratic costs, analytical results showed that time-delays have stabilizing effect on price dynamics. That is, time-delays serve to stabilize markets that are otherwise unstable (in the absence of time-delays).

Consistent with chaotic dynamics, the dominant root of the underlying system, comprising of complex eigenvalues, crosses the boundary of the unit circle as technological efficiency (k_1) and sensitivity to downstream prices (b) cross a critical threshold. Numerical experiments under judicious model calibrations corroborate this conclusion. First, simulated price trajectories exhibit bounded, quasi-periodic behavior in time-space, characterized by non-periodic, dense orbits in phase-space. Second, bifurcation diagrams illustrate that the onset of chaos takes place via Hopf bifurcation. Third, calculations of maximal Lyapunov exponent reveal sensitive dependence on initial conditions. Lastly, the underlying model is able to reproduce the stylized features of observed price data from the Pakistan poultry sector. In doing so, the paper addresses a major criticism of the theory of endogenous price fluctuations, negatively correlated prices. These findings have important ramification vis-à-vis the literature on price fluctuations of agricultural products.

From a policy perspective, the aforementioned analysis demonstrates that vertical linkages and the associated time-delays have profound effects on the dynamics of agricultural product prices. Moreover, endogenous, chaotic price fluctuations arising due to the interactions between vertically linked cobweb markets are not a borderline case, but rather occur over an empirically relevant range of model parameters. Of course, government interventions are particularly useful mechanisms to mitigate endogenous price fluctuations. However, policymakers need to be cognizant of the effects of vertical linkages between upstream and downstream sector at the time of designing policy interventions in a given agricultural market. Lastly, increasing technological efficiency combined with the existing low sensitivity to prices of agricultural products (especially chicken), may explain the increasingly unpredictable behavior of poultry prices over the past decade in less developed countries. A transition from live-bird markets to frozen bird markets is one way to minimize these price fluctuations. Similarly, a change in the organization of production is also required, whereby value chains move progressively towards full vertical integration.

References

Baak, S.J.1999.Tests for Bounded Rationality with a Linear Dynamic Model Distorted by Heterogeneous Expectations. *Journal of Economic Dynamics and Control* 23(9):1517-1543.

Boussard, J-M. 1996. When Risk Generates Chaos. *Journal of Economic Behavior and Organization*, Elsevier 29(3):433-446.

Brock, W.A. and C.H. Hommes. 1997. A Rational Route to Randomness. *Econometrica*. 65:1059–1160.

Brock, W.A., P.D.E Dindo, and C.H. Hommes. 2007. Adaptive Rational Equilibrium with Forward Looking Agents. *Int. J. Econ. Theory* 2:241–278.

Brock, W.A.1986. Distinguishing Random and Deterministic Systems. *Journal of Economic Theory* 40(1):68-95.

Chavas, J-P. 1999a. On Dynamic Arbitrage Pricing and Information: The Case of the US Broiler Sector. *European Review of Agricultural Economics* 26(4):493-510.

Chavas, J-P.1999b. On the Economic Rationality of Market Participants: The Case of Expectations in the U.S. Pork Market. *Journal of Agricultural and Resource Economics*. 24(1).

Chavas, J-P. 2000. On Information and Market Dynamics: The Case of the U.S. Beef Market. *Journal of Economic Dynamics and Control* 24(5-7):833-853.

Cafiero, C., B.D Wright, A. Sarris, and D. Hallam. 2006. Is the Storage Model a ‘Closed’ Empirical Issue? The Empirical Ability of the Storage Model to Explain Price Dynamics. *Agricultural Commodity Markets and Trade. New Approaches to Analyzing Market Structure and Instability 162*(89):114.

Campbell, S.A.2007. Time Delays in Neural Systems. In *Handbook of Brain Connectivity,* V.K. Jirsa and A.R McIntosh (eds.). Springer-Verlag.

Chaudhry, M. I., A. Katchova, and M.J Miranda. 2016. Examining Pricing Mechanics in the Poultry Value Chain: Empirical Evidence from Pakistan. Paper presented at the Annual meetings of the *Agricultural and Applied Economics Association,* July 31-August 2, 2016, Boston, Massachusetts (No. 235953).

Celso, G., E. Ott, and J.A. Yorke.1987. Chaos, Strange Attractors, and Fractal Basin Boundaries in Nonlinear Dynamics. *Science* 238(4827):632–638.

Chiarella, C. 1988.The Cobweb Model: Its Instability and the Onset of Chaos, *Economic Modeling* 5:377-384.

Deaton, A., and G. Laroque.1992. On the Behavior of Commodity Prices. *Rev. Econ. Stud.* 59:1–23.

Dieci, R., and F. Westerhoff, 2009. Stability Analysis of a Cobweb Model with Market Interactions. *Applied Mathematics and Computation*, 215(6):2011-2023.

Dieci, R., and F. Westerhoff. 2010. Interacting Cobweb Markets. *Journal of Economic Behavior and Organization* 75(3):461-481.

Emilia, F.2014. *Introduction to Time-Delay Systems: Analysis and Control*. Springer.

Ezekiel M. 1938. The Cobweb Theorem. *Quarterly Journal of Economics*52:255–280.

Farrell, D. 2013. *The Role of Poultry in Human Nutrition*, Food and Agricultural Organization of the United Nations.

Finkenstadt B., 1995. Nonlinear Dynamics in Economics: A Theoretical and Statistical Approach to Agricultural Markets. *Lecture Notes in Economics and Mathematical Systems* no. 426, Springer, Berlin, Germany.

Government of Pakistan.2014. Economic Survey of Pakistan. *Ministry of Finance, Government of Pakistan*.

Hale J.K., E.F., Infante and F.S.P Tsen. 1985. Stability in Linear Delay Equations. *Journal of Mathematical Analysis and Applications*105:533-555.

Hassan, S.Z. 2004. On Delayed Dynamical Duopoly. *Applied Mathematics and Computation* 151(1):275-286.

Heemeijer, P., C. Hommes, J. Sonnemans, and J. Tuinstra. 2009. Price Stability and Volatility in Markets with Positive and Negative Expectations Feedback: An Experimental Investigation. *Journal of Economic Dynamics and Control* 33(5):1052-1072.

Hommes, C.H. 1994. Dynamics of the Cobweb Model with Adaptive Expectations and Nonlinear Supply and Demand. *Journal of Economic Behavior and Organization*24:315–335.

Hommes, C.H. 1998.On the Consistency of Backward-Looking Expectations: The Case of the Cobweb. *Journal of Economic Behavior and Organization*33:333–362.

Hommes, C.H., 2013. *Behavioral Rationality and Heterogeneous Agents in Complex Economic Systems.* Cambridge University Press.

Hommes, C.H., andJ.B. Rosser. 2001. Consistent Expectations Equilibria and Complex Dynamics in Renewable Resource Markets. *Macroeconomic Dynamics 5*(2):180-203.

Hommes, C.H., J. Sonnemans, J.Tuinstra, and H. van de Velden. 2007. Learning in Cobweb Experiments. *Macroeconomic Dynamics* 11(S1):8-33.

Hommes, C.H.1991. Adaptive Learning and Roads to Chaos: The Case of the Cobweb. *Economics Letters* 36:127-132.

Jensen, R.V., and R. Urban.1984. Chaotic Price Behavior in a Non-Linear Cobweb Model. *Economics Letters* 15(3):235-240.

Kubin, I., andL. Gardini. 2013. Border Collision Bifurcations in Boom and Bust Cycles. *Journal of Evolutionary Economics 23*(4):811-829.

Lucas, R.E. 1980. Methods and Problems in Business Cycle Theory. *Journal of Money, Credit and Banking* 12(4):696-715.

Lundberg, L., E. Jonson, K. Lindgren, D. Bryngelsson, and V. Verendel, 2015. A Cobweb Model of Land-Use Competition between Food and Bioenergy Crops. *Journal of Economic Dynamics and Control*53:1-14.

Mitra, S.,and J. Boussard.2012. A Simple Model of Endogenous Agricultural Commodity Price Fluctuations with Storage. *Agricultural Economics*, 43(1):1-15.

Nerlove M. 1958. Adaptive Expectations and Cobweb Phenomenon. *Quarterly Journal of Economics*72:227–240.

Neusser, K. 2015. *Difference Equations for Economists.*

Onozaki, T., G. Sieg, and M. Yokoo. 2000. Complex Dynamics in a Cobweb Model with Adaptive Production Adjustment. *Journal of Economic Behavior and Organization* 41:101–115.

Puu, T. 2013. *Attractors, Bifurcations, and Chaos: Nonlinear Phenomena in Economics*. Springer Science and Business Media.

Haq, S. 2014. Poultry Sector is Spreading its Wings. *The Express Tribune-Business.*accessed from http://www.thepoultrysite.com/poultrynews/32813/pakistan-poultry-sector-spreading-its-wings/

Sonnemans, J., C.H. Hommes, J. Tuinstra, and H. van de Velden. 2004. The Instability of a Heterogeneous Cobweb Economy: A Strategy Experiment on Expectation Formation. *Journal of Economic Behavior and Organization* 54:453-481.

Tomek, W.G., andK.L. Robinson. 2003. *Agricultural Prices*. Ithaca, NY: Cornell University Press.

Westerhoff, F., and C. Wieland. 2010. A Behavioral Cobweb-Like Commodity Market Model with Heterogeneous Speculators. *Economic Modeling* 27(5):1136-1143.

Vas L. 2016. Modeling with Systems of Difference and Differential Equations. *Mathematical Modeling.* accessed from http://www.usciences.edu/~lvas/math422/systems.pdf

Yassen, M.T., and H.N. Agiza.2003.Analysis of a Duopoly Game with Delayed Bounded Rationality. *Applied Mathematics and Computation*138(2):387-402.

Zavarei, M.M. and M. Jamshidi. 1987.*Time-Delay Systems Analysis, Optimization and Applications.* North Holland, Amsterdam.

Appendix 1: Proof of Existence and Uniqueness of Equilibrium State

In steady state (regardless of $\Delta t = 1$ or $\Delta t = 3$), $p_t^B = p_{t-\tau}^B = p^B$ and $\omega_t^C = \omega_{t-\tau}^C = \omega^c$ for $\forall\, \tau$, where τ is the set of integers. Substituting this into the system of equations, we get:

$$p^B = \frac{a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 k_1}{bN_3} (k_1 p^B - \omega^c)^{\frac{1}{\beta-1}} \qquad (1)$$

$$\omega^c = k_1 p^B - \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}} \qquad (2)$$

From equation-2 we have:

$$p^B = \frac{1}{k_1}\left(\omega^c + \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}}\right)$$

Substituting this result into equation-1and simplifying yields:

$$\omega^c + \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}} = \frac{k_1 a}{b} - \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 {k_1}^2}{bN_3} \left(\beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}}\right)^{\frac{1}{\beta-1}}$$

Collecting all terms with ω^cin the left-hand-side:

$$\omega^c + \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}} + \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \frac{N_2 {k_1}^2}{bN_3} \left(\beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}}\right)^{\frac{1}{\beta-1}} = \frac{k_1 a}{b}$$

After some algebra, we get:

$$\omega^c + \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1} \left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} (k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}} + \left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}} \beta^{\frac{1}{\beta-1}} \left(\frac{N_2 {k_1}^2}{bN_3}\right)\left(\frac{N_1 k_2}{N_2}\right)\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} (k_2\omega^c - e)^{\frac{1}{\alpha-1}} = \frac{k_1 a}{b}$$

Some further simplifications yield:

$$\omega^c + \beta\left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1}(k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}} + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{N_1 {k_1}^2 k_2}{bN_3}\right)(k_2\omega^c - e)^{\frac{1}{\alpha-1}} = \frac{k_1 a}{b}$$

Let us define the left-hand-side as $f(\omega^c)$, from the chick farmers profit maximization problems we know that $k_2\omega^c - e > 0$, otherwise chick farms simply shutdown. It is trivial to show that $f(\omega^c)$ is well-defined over the interval $\left[\frac{e}{k_2}, \infty\right]$ and hence continuous.

Now we show that $f'(\omega^c) > 0$ i.e. $f'(\omega^c) = 1 + \beta\left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1}\left(\frac{\beta-1}{\alpha-1}\right)k_2(k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}-1} + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{N_1 {k_1}^2 k_2}{bN_3}\right)\left(\frac{1}{\alpha-1}\right)k_2(k_2\omega^c - e)^{\frac{1}{\alpha-1}-1}$ is positively valued because $k_2\omega^c - e > 0$ by the chick farmers profit maximization problem, while, β and α are greater than 1 by the assumed (strict) convexity of the cost functions, and all other parameters are positive numbers. Therefore, we know that$f(\omega^c)$ is a continuous, increasing function defined on the interval $\left[\frac{e}{k_2}, \infty\right]$ and the range of $f(\omega^c)$ is $[\delta, \infty]$where $\delta = f\left(\frac{e}{k_2}\right) > \frac{e}{k_2} > 0$.

Now we can examine the roots of the equation: $f(\omega^c) = \frac{k_1 a}{b}$. First, as far as existence is concerned, note that since $f(\omega^c)$ is a continuously defined on every point in the interval $\left[\frac{e}{k_2}, \infty\right]$, then f will take on every value between $f\left(\frac{e}{k_2}\right)$ and $f(\infty)$. Therefore, by the intermediate value theorem there exist some values of ω^c between $\left[\frac{e}{k_2}, \infty\right]$ such $f(\omega^c) = \frac{k_1 a}{b}$ as long as $\delta < \frac{k_1 a}{b} < \infty$ i.e. $\frac{k_1 a}{b}$ is in the range of f. Second, we know this root is unique because $f(\omega^c)$ is a strictly increasing function, i.e. $f(\omega^c)$ crosses the horizontal line $\frac{k_1 a}{b}$ at a single point. Of course, this conclusion is not trivial because a nonlinear equation may have zero, one or infinite fixed points, but a linear system has always one or no fixed points.

Given that the root of $f(\omega^c) = \frac{k_1 a}{b}$ is positively valued and unique, we can numerically solve for ω^c and thereafter back out the corresponding value of p^B using the fact that $p^B = \frac{1}{k_1}\left(\omega^c + \beta\left(\frac{N_1 k_2}{N_2}\right)^{\beta-1}\left(\frac{1}{\alpha}\right)^{\frac{\beta-1}{\alpha-1}}(k_2\omega^c - e)^{\frac{\beta-1}{\alpha-1}}\right)$. It is easy to show that $p^B > \omega^c$ in the steady state becausethe second term in the brackets is a positive number since $k_2\omega^c - e > 0$ by the chick farmers profit maximization problem, $0 < k_1 < 1$ and all other parameters are positive numbers.

Appendix 2: Analysis of Local Stability Using State-Space Augmentation

The linear approximation of the underlying system in the neighborhood of the equilibrium state (p^B, ω^c) is given by:

$$\begin{cases} p_t^B = \left(\left. \dfrac{\partial p_t^B}{\partial \omega_{t-2}^C} \right|_{p^B \omega^c} \right) \omega_{t-2}^C + \left(\left. \dfrac{\partial p_t^B}{\partial p_{t-2}^B} \right|_{p^B \omega^c} \right) p_{t-2}^B \\ \omega_t^C = \left(\left. \dfrac{\partial \omega_t^C}{\partial \omega_{t-1}^C} \right|_{p^B \omega^c} \right) \omega_{t-1}^C + \left(\left. \dfrac{\partial \omega_t^C}{\partial p_{t-1}^B} \right|_{p^B \omega^c} \right) p_{t-1}^B \end{cases}$$

Constants A, B, C and D represent the respective partial derivatives evaluated at the fixed point of the system:

$$A : \left. \frac{\partial p_t^B}{\partial p_{t-2}^B} \right|_{p^B \omega^c} = -\frac{1}{\beta - 1} \left(\frac{1}{\beta} \right)^{\frac{1}{\beta - 1}} \frac{N_2 {k_1}^2}{b N_3} (k_1 p^B - \omega^c)^{\frac{1}{\beta - 1} - 1} < 0$$

$B : \left. \frac{\partial p_t^B}{\partial \omega_{t-2}^C} \right|_{p^B \omega^c} = \frac{1}{\beta - 1} \left(\frac{1}{\beta} \right)^{\frac{1}{\beta - 1}} \frac{N_2 k_1}{b N_3} (k_1 p^B - \omega^c)^{\frac{1}{\beta - 1} - 1} > 0$ (note that $a = -k_1 b$)

$$C : \left. \frac{\partial \omega_t^C}{\partial p_{t-1}^B} \right|_{p^B \omega^c} = k_1 > 0$$

$$D : \left. \frac{\partial \omega_t^C}{\partial \omega_{t-1}^C} \right|_{p^B \omega^c} = -\beta k_2 \left(\frac{\beta - 1}{\alpha - 1} \right) \left(\frac{N_1 k_2}{N_2} \right)^{\beta - 1} \left(\frac{1}{\alpha} \right)^{\frac{\beta - 1}{\alpha - 1}} (k_2 \omega^c - e)^{\frac{\beta - 1}{\alpha - 1} - 1} < 0$$

The linearized system can now be expressed as:

$$\begin{cases} p_t^B = B \omega_{t-2}^C + A p_{t-2}^B \\ \omega_t^C = D \omega_{t-1}^C + C p_{t-1}^B \end{cases}$$

Now we use the state-space augmentation method to recast this time-delay system into a higher-order system comprising of 1st order difference equations only. This approach is based upon the fact that a time-delay difference equation is equivalent to a higher order difference equation[23]. And given an arbitrary n^{th} order difference equation, it is trivial to produce a system of n first order difference equations by defining $n - 1$ new variables for each higher order difference term.

It is easy to see that the linearized system is a 2nd order system with two state variables. We define two (or, in general, $2(n-1)$, where n is the number of state variables in the system) variables to represent each higher order difference term. Let $x_t^1 = p_{t-1}^B$ and $y_t^1 = \omega_{t-1}^C$, or more generally $x_t^i = p_{t-i}^B$ and $y_t^i = \omega_{t-i}^C$ for $i = 1,2$. Substituting these definitions into the linearization of the time-delay system above gives us a new 4x4 system of first order difference equations:

[23] The order of a system of difference equation is defined as the maximal difference between the highest and lowest time indexes for a given variable in any system of difference equations.

$$\begin{bmatrix} p_t^B \\ \omega_t^C \\ x_t^1 \\ y_t^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & A & B \\ C & D & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{t-1}^B \\ \omega_{t-1}^C \\ x_{t-1}^1 \\ y_{t-1}^1 \end{bmatrix}$$

It is easy to verify that the first two rows of this transition matrix are equivalent to the original relationships shown in the linearization of the time-delay system, while the bottom two rows are merely first-order identities of new variables defined above. For notational simplicity, let X_t denote the left-hand-side vector of state-variables and J represent the transition matrix, then note that the model can be succinctly written as a first order system:$X_t = JX_{t-1}$. It is well known, that eigenvalues of the matrix J will determine the local behavior of the original time-delay system (see Neusser 2015 for why this is so).

Using Laplace expansion along the last row and after some tedious algebra, we get the following expression for the characteristic polynomial, the roots of which can be obtained by appeal to standard numerical methods:

$$\lambda^4 - D\lambda^3 - A\lambda^2 + \lambda A(D+1)$$

Aside, note that in the equivalent configuration with $\Delta t = 1$ week, the same process i.e. linearization around equilibrium followed by state-space augmentation can be employed to show that the resulting transition matrix has a dimension of 12 x 12.The definition of first-order approximation around the equilibrium state are identical but with different time-indexes. Let$x_t^1 = p_{t-1}^B$and $y_t^1 = \omega_{t-1}^C$, or more generally $x_t^i = p_{t-i}^B$ and $y_t^i = \omega_{t-i}^C$ for $i = 1,2,3,4,5$. The state-space augmentation approach yields:

$$\begin{bmatrix} p_t^B \\ \omega_t^C \\ x_t^1 \\ y_t^1 \\ x_t^2 \\ y_t^2 \\ x_t^3 \\ y_t^3 \\ x_t^4 \\ y_t^4 \\ x_t^5 \\ y_t^5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & C & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{t-1}^B \\ \omega_{t-1}^C \\ x_{t-1}^1 \\ y_{t-1}^1 \\ x_{t-1}^2 \\ y_{t-1}^2 \\ x_{t-1}^3 \\ y_{t-1}^3 \\ x_{t-1}^4 \\ y_{t-1}^4 \\ x_{t-1}^5 \\ y_{t-1}^5 \end{bmatrix}$$

Clearly, it is not feasible to derive the associated characteristic equation. Although computationally burdensome, numerical methods show that the behavior of the modulus of dominant eigenvalue with respect to parameters of interest in both cases (i.e. $\Delta t = 1$ week and $\Delta t = 3$ weeks) is qualitatively identical.

Appendix 3: Stability Independent of Time-Delays in the case of Quadratic Costs

First note that in the special case of quadratic costs, i.e., $\alpha = \beta = 2$, the underlying time-delay system is rendered linear:

$$p_t^B = \frac{a}{b} - \frac{N_2 {k_1}^2}{2bN_3} p_{t-6}^B + \frac{N_2 k_1}{2bN_3} \omega_{t-6}^C$$
$$\omega_t^C = k_1 p_{t-3}^B - \frac{N_1 {k_2}^2}{N_2} \omega_{t-3}^C + \frac{N_1 k_2}{N_2} e$$

The discrete time analogue of the Hale et al. (1985) theorem states that if x represents a vector of endogenous state variables with fixed time-delay d and coefficient matrix A and A_d, such that the resulting system of linear time-delay difference equations is given by$\Delta x_t = A x_{t-1} + A_d x_{(t-d)}$[24], then the system is asymptotically stable, independent of delay if $Re\,|\lambda(A + A_d)| < 1$.Applying the theorem to the underlying linear time-delay system above, we get:

$$\begin{bmatrix} \Delta p_t^B \\ \Delta \omega_t^C \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_{t-1}^B \\ \omega_{t-1}^C \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & -\frac{N_1 {k_2}^2}{N_2} \end{bmatrix} \begin{bmatrix} p_{t-3}^B \\ \omega_{t-3}^C \end{bmatrix} + \begin{bmatrix} -\frac{N_2 {k_1}^2}{2bN_3} & \frac{N_2 k_1}{2bN_3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{t-6}^B \\ \omega_{t-6}^C \end{bmatrix} + \begin{bmatrix} \frac{a}{b} \\ \frac{N_1 k_2}{N_2} e \end{bmatrix}$$

Following Hale et al. (1989), we need to look at the eigenvalues of the following matrix in order to determine the asymptotic stability of the underlying linear time-delay system independent of time delay:

$$\lambda(A + A_d) = \begin{bmatrix} -1 - \frac{N_2 {k_1}^2}{2bN_3} & \frac{N_2 k_1}{2bN_3} \\ k_1 & -1 - \frac{N_1 {k_2}^2}{N_2} \end{bmatrix}$$

The corresponding characteristic equation is:

$$\left(1 + \frac{N_2 {k_1}^2}{2bN_3} + \lambda\right)\left(1 + \frac{N_1 {k_2}^2}{N_2} + \lambda\right) - \frac{N_2 {k_1}^2}{2bN_3} = 0$$

Multiplying out and collecting terms we get:

[24] The original theorem was derived in continuous-time for a system retarded differential equations given by: $\dot{x} = Ax + A_d x_{(t-d)}$. However, under standard normality conditions, it is easy to show that the theorem also holds for linear time-delay difference equations with a fixed delay, as retarded differential equations and linear time-delay difference equations with fixed delay are equivalent in aspects expect for time configuration and one can be readily converted into the other, see Vas (2016).

$$\lambda^2 + \lambda\left(2 + \frac{N_2 k_1^{\ 2}}{2bN_3} + \frac{N_1 k_2^{\ 2}}{N_2}\right) + 1 + \frac{N_1 k_2^{\ 2}}{N_2} + \frac{N_1 k_1^{\ 2} k_2^{\ 2}}{2bN_3} = 0$$

Now, using the quadratic root formula: $\frac{-b}{2a}\frac{\pm\sqrt{(b^2-4ac)}}{2a}$, where$b = 2 + \frac{N_2 k_1^{\ 2}}{2bN_3} + \frac{N_1 k_2^{\ 2}}{N_2}$, $c = 1 + \frac{N_1 k_2^{\ 2}}{N_2} + \frac{N_1 k_1^{\ 2} k_2^{\ 2}}{2bN_3}$ and $a = 1$,note that $\left|\frac{-b}{2a}\right| > 1$, since $\frac{N_1 k_2^{\ 2}}{N_2}$ and $\frac{N_2 k_1^{\ 2}}{2bN_3}$ are non-zero positive numbers by definition of the parameters. This is sufficient to prove that the magnitude of the dominant root is greater than 1. Because, in case the of complex conjugates,i.e.,$r + iw$if$b^2 - 4ac < 0$, it is straightforward to notice that the magnitude of real part is greater than 1 as $r = \frac{-b}{2a}$, indicative of (asymptotic) divergence. While in the case of $b^2 - 4ac \geq 0$, one of the real root is necessarily greater than 1 in magnitude, i.e.,$\left|-\frac{b}{2a} - \frac{\sqrt{(b^2-4ac)}}{2a}\right|$ since $\frac{b}{2a} > 1$.

Nevertheless, for sake of completeness, after some tedious algebra and simplifying the expression of $(b^2 - 4ac)$ we get: $\frac{N_2^{\ 2} k_1^{\ 4}}{4b^2 N_3^{\ 2}} + \frac{N_1^{\ 2} k_2^{\ 2}}{N_2^{\ 2}} + \frac{2k_1^{\ 2}(2N_2 - N_1 k_2^{\ 2})}{2bN_3}$. This expression is strictly positive by definition of the parameters[25]. We know with certainty that that absolute value of $\lambda_1 = -\frac{b}{2a} - \frac{\sqrt{(b^2-4ac)}}{2a}$ is greater than 1, since both $\frac{b}{2a}$ and $\frac{\sqrt{(b^2-4ac)}}{2a}$ are positive, while $\left|\frac{b}{2a}\right|$>1 by definition. For our purposes it suffices to note that $|\lambda_1|$>$|\lambda_2|$ where $\lambda_2 = -\frac{b}{2a} + \frac{\sqrt{(b^2-4ac)}}{2a}$. The asymptotic behavior of the system is determined by the dominant or largest eigenvalue and it is well known that the underlying system is asymptotically unstable if the absolute value of the dominant eigenvalue is greater than 1. Therefore, we conclude that the underlying system is asymptotically unstable independent of time-delays. Note that in addition to $|\lambda_1|$>1, λ_1 is also negative, consistent with oscillatory diverging behavior, while the sign of λ_2 depends on the parameter values. Likewise, depending on the parameter values either $|\lambda_1|$>1>$|\lambda_2|$ which corresponds to a saddle point or an unstable explosive system $|\lambda_1|$>$|\lambda_2| > 1$.

In order to get a better idea of the behavior of the linear system without time-delays, we compute eigenvalues using model calibrations identical to the baseline case (expect $\alpha = \beta = 2$) i.e. $k_1 = 0.93$, $k_2 = 0.95$, $b = 6.5$,$a = 1000$, $e = 3$, $N_1 = N_2 = N_3 = 100$. The resulting eigenvalues are $\lambda_1 = -1.98$ and $\lambda_2 = -0.99$. This is consistent with a saddle-point, whereby the system is best characterized by oscillatory (asymptotic) divergence away from the equilibrium or explosion in the direction of λ_1 and oscillatory convergence towards equilibrium in the direction of λ_2. Therefore, independent of time-delays, the underlying price dynamic (in the case of quadratic costs) is asymptotically unstable due to the effect of the dominant eigenvalue[26].

Using state-space augmentation methods described in appendix-2, we compute eigenvalues under identical calibrations but with the effects of time-delays. The (non-zero) eigenvalues of the resulting linear system were $\lambda_{1,2} = 0.4662 \pm 0.8075i$, $\lambda_{3,4} = 0.2636 \pm 0.4566i$, $\lambda_{5,6} = -0.1897 \pm 0.3286i$, $\lambda_7 = -0.9324$, $\lambda_8 = 0.3794$, $\lambda_9 = -0.5272$. It is easy to see that the an

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[25] As mentioned before, given this is a short run price dynamic model,N_1,$N_2$and N_3 are simply scale parameters and assumed to be roughly equal. Since $0 < k_2 < 1$, then $2N_2 - N_1 k_2^{\ 2} > 0$.

[26] λ_2 is approximately equal to -1, indicative of the borderline case of perpetual oscillations around the equilibrium in linear models.

otherwise identical linear system but with time-delays is asymptotically stable since the modulus of all eigenvalues is less than 1. This is consistent with statement of proposition 2 i.e. time-delays stabilize an otherwise unstable price dynamic.