## Deriving the Marginal Rate of Transformation

Firms hire factors of production up to point where value of marginal product equals factor price, i.e.,

$$
\begin{array}{ll}
\mathbf{P}_{\mathbf{X}} \mathbf{M} \mathbf{P}_{\mathrm{LX}}=\mathbf{w} & \mathbf{P}_{\mathbf{X}} \mathbf{M} \mathbf{P}_{K X}=\mathbf{r}  \tag{1}\\
\mathbf{P}_{\mathbf{Y}} \mathbf{M} \mathbf{P}_{\mathrm{LY}}=\mathbf{w} & \mathbf{P}_{\mathbf{Y}} \mathbf{M} \mathbf{P}_{K Y}=\mathbf{r}
\end{array}
$$

Divide through:

$$
\begin{equation*}
\frac{\mathbf{P}_{\mathrm{X}} \mathbf{M} \mathbf{P}_{\mathrm{LX}}}{\mathbf{P}_{\mathbf{Y}} \mathbf{M} \mathbf{P}_{\mathrm{LY}}}=\frac{\mathbf{w}}{\mathbf{w}} \quad \frac{\mathbf{P}_{\mathrm{X}} \mathbf{M} \mathbf{P}_{\mathrm{KX}}}{\mathbf{P}_{\mathbf{Y}} \mathbf{M} \mathbf{P}_{\mathrm{KY}}}=\frac{\mathbf{r}}{\mathbf{r}} \tag{2}
\end{equation*}
$$

Rearrange (2):

$$
\begin{equation*}
\frac{\mathbf{P}_{\mathbf{X}}}{\mathbf{P}_{\mathbf{V}}}=\frac{\mathbf{M} \mathbf{P}_{\mathrm{LY}}}{\mathbf{M P}_{\mathrm{LX}}}=\frac{\mathbf{M} \mathbf{P}_{\mathrm{KY}}}{\mathbf{M P}_{\mathrm{KX}}} \tag{3}
\end{equation*}
$$

Production functions for $X$ and $Y$ are:

$$
\begin{align*}
\mathbf{X} & =\mathbf{f}(\mathbf{K}, \mathbf{L})  \tag{4}\\
\mathbf{Y} & =\mathbf{g}(\mathbf{K}, \mathbf{L})
\end{align*}
$$

Partial derivatives of (4) with respect to $K$ and $L$ give marginal products of capital and labor, i.e.,

$$
\begin{array}{ll}
\partial \mathbf{X} / \partial \mathbf{K}_{\mathbf{X}}=\mathbf{M} \mathbf{P}_{\mathbf{K X}}, & \partial \mathbf{X} / \partial \mathbf{L}_{\mathbf{X}}=\mathbf{M} \mathbf{P}_{\mathbf{L X}}  \tag{5}\\
\partial \mathbf{Y} / \partial \mathbf{K}_{\mathbf{Y}}=\mathbf{M P}_{\mathbf{K Y}}, & \partial \mathbf{Y} / \partial \mathbf{L}_{\mathbf{Y}}=\mathbf{M P}_{\mathbf{L Y}}
\end{array}
$$

Substituting (5) into (3):

$$
\begin{equation*}
\frac{\mathbf{P}_{\mathbf{X}}}{\mathbf{P}_{\mathbf{Y}}}=\frac{\partial \mathbf{Y} / \partial \mathbf{L}_{\mathbf{Y}}}{\partial \mathbf{X} / \partial \mathbf{L}_{\mathbf{x}}}=\frac{\partial \mathbf{Y} / \partial \mathbf{K}_{\mathbf{Y}}}{\partial \mathbf{X} / \partial \mathbf{K}_{\mathbf{x}}} \tag{6}
\end{equation*}
$$

Given factors of production are fixed in supply, then $\partial \mathbf{L}_{\mathbf{X}}=-\partial \mathbf{L}_{\mathbf{Y}}$ and $\partial \mathbf{K}_{\mathbf{x}}=-\partial \mathbf{K}_{\mathrm{Y}}$, therefore:

$$
\begin{equation*}
\frac{\mathbf{P}_{\mathbf{X}}}{\mathbf{P}_{\mathbf{Y}}}=-\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}=\mathbf{M R T} \tag{7}
\end{equation*}
$$

