

Effective Risk Aversion and the Demand for Savings and Insurance

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Abstract

We examine the tradeoffs and complementarities that exist among saving, borrowing, and insurance in managing generic income risk. We depart from conventional static Von Neumann-Morgenstern Expected Utility theory by couching the insurance adoption decision in a formal dynamic framework that explicitly allows for the use of alternative financial instruments to manage risk. We find that the standard results of expected utility theory do not hold when agents may borrow or save. Access to savings and borrowing “crowds out”, that is, reduces the demand for, insurance. Agents who have access to borrowing and saving: 1) do not fully insure at actuarially favorable premium rates; 2) are less likely to insure at loaded “market” premiums rates; and 3) may not insure at all, even if premium rates are subsidized, if they are temporarily poor. We show that the demand for insurance depends, not simply the curvature of the utility of consumption function, but rather, more generally, on the curvature of the value function, which we refer to as the “effective risk aversion. Effective

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risk aversion is heavily influenced by market conditions other than an individual's primitive attitude toward risk, most notably the degree of access to borrowing and savings.

Introduction

Methods for managing agricultural production risk have been the subject of an extensive body of research that has appeared in the agricultural economics literature over the past forty years, with considerable attention being devoted to the optimal use of agricultural crop insurance.¹ Most theoretical findings regarding the benefits of agricultural crop insurance employ static Von Neumann-Morgenstern expected utility models in which an agricultural producer's attitude toward risk is represented by a utility of money function whose concave curvature captures the producer's aversion to risk. Typically, such models predict that a producer can enjoy substantial benefits from purchasing crop insurance, even when the premium exceeds the expected indemnity.

It is a well-established empirical fact, however, that agricultural producers employ crop insurance at rates that are substantially less than those predicted or prescribed by static expected utility models, even when the insurance contracts embody subsidies (see Knight and Coble 1997). A widespread view among agricultural risk management researchers, policy makers, and extension agents is that the discrepancy between theory and practice is largely explained by producers' ignorance on how best to employ crop insurance in their production and marketing plans, prompting calls for aggressive public outreach and extension programs to educate agricultural producers on how to improve their financial welfare through proper use of crop insurance.

The discrepancy between theory and practice, however, may be due more to an incomplete theory of decision making under uncertainty than to flawed agricultural producer practices and perceptions. Agricultural risk management studies employing static, expected utility models are unable to ade-

¹Between 1970 and 2011, the American Journal of Agricultural Economics alone published 649 articles containing "risk" in the title and 137 articles containing "risk" and "insurance" in the title.

quately capture effective available alternatives to crop insurance for managing farm level risk, most notably the use of saving, borrowing, and investment, which are inherently dynamic in nature. Consequently, conventional static models over-state the benefits of crop insurance and cannot adequately inform us on optimal uses of crop insurance and how its benefits might depend on a producer's net worth, liquidity, and access to and cost of credit.

In this paper, we systematically explore how access to credit and savings impacts the benefits and optimal uses of agricultural crop insurance. Our analysis is grounded in formal theoretical models of dynamically optimizing agents that have access not only to crop insurance, but also to financial instruments that allow the agent to build financial reserves through savings. We begin by exploring the interrelations between saving and insurance using a two-period model for which analytic solutions exist. In the context of this model, we establish that the demand for and benefits of crop insurance rely heavily on a producer's financial reserves and his/her access to credit and savings. In particular, savings and insurance are substitutes, with the demand for insurance declining with interest rates on savings, and the demand for net savings increasing with insurance premium rates. Our analysis also reveals that the demand for insurance is more a function of the producer's financial reserves than the risk aversion embodied in his/her utility of consumption function, and leads us to conclude that insurance is something the poor cannot afford and the rich do not need. None of these results emerge from the standard static model of decision-making under uncertainty that is used almost exclusively in published theoretical and empirical work on crop insurance demand.

We then turn to an infinite-horizon model of crop insurance demand when producers are able to save. This model, due to its inherent complexity, cannot be solved analytically, but may be solved and analyzed using computational techniques. In the context of a fully dynamic model, the demand for crop insurance will depend primarily on the curvature of the producer's value function, not simply the curvature of his/her utility of consumption function. The curvature of the value function, moreover, is endogenous to the model, being determined not only by the agent's utility of consumption but also access to savings as indicated by the interest rates at which he/she can save. The model yields additional implications that do not emerge from conventional static expected utility models or a two-period dynamic model. In particular,

in a fully dynamic setting, a producer's value function in the presence of borrowing and saving exhibits less curvature, and thus embodies less effective risk aversion, than the producer's utility of consumption function. As such, the short- and long-run demand for crop insurance with access to savings is less than predicted by the two-period model. The curvature of the value function, moreover, can be reversed at low levels of wealth if bankruptcy and loan restructuring options exist, turning the producer into what in the context of static expected utility would be considered a "risk-lover" who would have no demand for crop insurance.

Our research answers questions of fundamental interest to researcher, practitioners, agricultural economists and policymakers interested in agricultural risk management not only in the developed countries, but also in the developing world. Our insights can help us explain the observed low demand for crop insurance in developed countries and shed light on the proper role for agricultural insurance in agricultural risk management. Our findings also inform the emerging debate among development economists on whether poor farmers in developing countries are better served by public programs that promote access to savings or by programs that promote access to crop or weather insurance.

Literature Review

Although the potential tradeoffs between savings and insurance have not received much attention from agricultural economists, the topic has been examined and documented in the general theoretical economics and insurance literatures. Ehrlich and Becker 1972 were first to develop a theory of decision making under uncertainty that allows for interaction between self-insurance, of which savings is one example, and market insurance, concluding that the two are substitutes. In their words: "optimal decisions about market insurance depend on the availability of ... other activities and should be viewed within the context of a more comprehensive insurance decision." The further point out that: "apparent attitudes toward risk are dependent on market opportunities, and real attitudes cannot easily be inferred from behavior."

Deaton 1991 analyzes optimal intertemporal consumption behavior of consumers who are able to borrow and save. He finds that when incomes exhibit low orders of autocorrelation over time, consumers are able to significantly smooth consumption over time by accumulating modest buffer stock of assets. In his model, however, the option to purchase insurance does not exist. Paxson 1992 and Morduch 1995 conclude that an individual's ability to save and borrow can greatly smooth consumption in the absence of alternative risk markets. Eisenhauer 1994 considers consumer credit as a source of risk financing and derives explicit conditions under which it is superior to insurance policies. He concludes that credit can be a viable alternative to insurance for financing low-severity household risks.

Gollier 1994 examines dynamic risk management strategies and concludes that risk-averse individuals will prefer self-insurance in the long run if the cost of insurance exceeds a certain critical value, suggesting that market insurance is most useful as a transitory strategy to protect capital and savings early in the life cycle. Gollier 2002 develops a theoretical foundation to self-insurance through intertemporal diversification. He finds that in a dynamic setting, attitudes toward wealth risk and consumption risk are not the same, and self-insurance becomes an effective substitute to costly external insurance.

Most recently, Gollier 2003 analyzes a simple life-cycle model based on Deaton 1991, Heaton and Lucas 1996, and Carroll 1997, in which consumers can follow a time-diversification (self-insurance) strategy by accumulating buffer stock wealth. He concludes that insurance would only be demanded for catastrophic risks or by individual that face liquidity constraints, and thus the added value of the insurance sector would be surprisingly low in such an economy.

Absorption of production and price shocks through some form of self-insurance is especially common in developing countries. Deaton 1992 analyzes how farmers in developing countries protect their living standards against fluctuations in income, emphasizing the role of credit markets in consumption smoothing. Hubbard, Skinner, and Zeldes 1995 employ a dynamic model with multiple sources of uncertainty to demonstrate how social insurance programs can discourage self-insurance and precautionary savings among poor households. Dercon 2002 provides a comprehensive review of recent literature on risk-management and risk-coping strategies of poor households,

identifies constraints on their effectiveness, and discusses policy options. He shows that the opportunities to use assets as insurance are limited by risk, lumpiness of assets, and entry constraints, and that informal risk-sharing provides only limited protection, leaving some of the poor exposed to very severe negative shocks.

Kurosaki and Fafchamps 2002 also review a body of research on the typical mechanisms of consumption smoothing, which include accumulating grain, livestock, and financial assets as a form of precautionary saving. They further refer to a body of literature that analyzes efficiency of these institutions and predicts that full insurance will not be achieved despite substantial consumption smoothing. They also find that the households adapt production practices in response to consumption price risk and warn that “empirical and theoretical work on risk should avoid putting an exclusive emphasis on yield and output price risk”.

Review of Static Model

We begin by briefly reviewing the demand for insurance in the static Von Neumann-Morgenstern expected utility model, which has been the bulwark of risk analysis in agriculture and other fields of economics.

Consider an agent endowed with predetermined initial wealth $w > 0$ who faces an uncertain income $\tilde{y} \geq 0$ and, additionally, an uncertain, but insurable loss $\tilde{l} \geq 0$ that is independent of income, with $E\tilde{l} > 0$. The agent may insure any portion $x \in [0, 1]$ of the loss at a premium rate $\pi > 0$.² That is, if the agent pays a premium $x\pi$, he receives an indemnity $x\tilde{l}$ if he experiences a loss of magnitude \tilde{l} . The agent chooses the coverage x that maximizes his expected utility of terminal wealth; that is, he solves:

$$\max_{0 \leq x \leq 1} U(x)$$

where

$$U(x) \equiv Eu(w + \tilde{y} - (1 - x)\tilde{l} - \pi x).$$

²In order to preclude the possibility of nonpositive terminal wealth, we further assume that $w + \underline{y} > \bar{l}$, where $\underline{y} \equiv \sup\{y \mid \Pr(\tilde{y} \leq y) = 0\}$ is the greatest lower bound on attainable income and $\bar{l} \equiv \inf\{l \mid \Pr(\tilde{l} \geq l) = 1\}$ is the least upper bound on attainable losses.

Here, u is the agent's utility of wealth, which is presumed to be twice continuously differentiable, strictly increasing, and strictly concave. Under these assumptions, U is twice continuously differentiable, strictly increasing, and strictly concave, implying that the optimal coverage x is well-defined and completely characterized by the Karush-Kuhn-Tucker conditions:

$$0 \leq x \leq 1, \quad x > 0 \implies U'(x) \geq 0, \quad x < 1 \implies U'(x) \leq 0.$$

Theorem 1. *The agent purchases coverage if, and only if,*

$$\pi < \pi^* \equiv E\lambda(\tilde{l})\tilde{l}.$$

where

$$\lambda(l) \equiv \frac{Eu'(w + \tilde{y} - l)}{Eu'(w + \tilde{y} - \tilde{l})}.$$

Moreover,

$$\pi^* > E\tilde{l}.$$

Proof. The Karush-Kuhn-Tucker conditions assure that the agent purchases coverage if, and only if,

$$U'(0) = E(\tilde{l} - \pi)u'(w + \tilde{y} - \tilde{l}) > 0.$$

The first part of the proposition follows directly with some simple algebraic manipulations. The second part of the proposition follows from the ‘‘covariance rule’’ (Gollier 2001, p. 94) and the facts that λ is strictly increasing, \tilde{l} is non-degenerate, and $E\lambda(\tilde{l}) = 1$. \square

Theorem 1 states that agent purchases coverage if, and only if, the premium rate π is less than the ‘‘risk adjusted’’ expected loss π^* , defined as the expectation of the loss \tilde{l} weighted by the relative marginal utility of wealth $\lambda(\tilde{l})$. The risk-adjusted expected loss π^* exceeds the expected loss $E\tilde{l}$ because the marginal utility of wealth is greater for large losses than for small losses. This implies that the agent will purchase coverage even if the premium rate exceeds the expected indemnity, provided the premium rate is not excessively high.

Theorem 2. *The agent will fully insure, that is, $x = 1$, if, and only if, the expected loss $E\tilde{l}$ equals or exceeds the premium rate π .*

Proof. Since \tilde{y} and \tilde{l} are independent,

$$U'(1) = E(\tilde{l} - \pi)u'(w + \tilde{y} - \pi) = (E\tilde{l} - \pi)Eu'(w + \tilde{y} - \pi).$$

Given that U' is strictly decreasing, the Karush-Kuhn-Tucker conditions guarantee that $x \geq 1 \iff U'(1) \geq 0 \iff E\tilde{l} \geq \pi$. \square

Theorem 3. *For $E\tilde{l} < \pi < \pi^*$, the optimal coverage x is a continuously differentiable function of the premium rate π .*

Proof. If $E\tilde{l} < \pi < \pi^*$, the agent will purchase coverage $x > 0$ such that $U'(x) = 0$. Thus, the Implicit Function Theorem guarantees that the optimal coverage $x(\pi)$ is a continuously differentiable function of the premium rate with derivative

$$x'(\pi) = \frac{x E(\tilde{l} - \pi)u''(w + \tilde{y} - (1 - x)\tilde{l} - \pi x) + Eu'(w + \tilde{y} - (1 - x)\tilde{l} - \pi x)}{E(\tilde{l} - \pi)^2 u''(w + \tilde{y} - (1 - x)\tilde{l} - \pi x)}.$$

It is not possible to sign this derivative globally without further assumptions. \square

Note 1. *At this point, I have not been able to sign this derivative globally. I should review Gollier for possible insights into how to do this*

Theorem 4. *Let $V(w)$ denote the maximum expected utility for a given wealth w , that is,*

$$V(w) = \max_{0 \leq x \leq 1} U(x, w)$$

where

$$U(x, w) \equiv Eu(w + \tilde{y} - (1 - x)\tilde{l} - \pi x).$$

Then V is continuously differentiable, strictly increasing, and strictly concave.

Proof. The continuous differentiability and the strict monotonicity of V follow directly from the Envelope Theorem, which guarantees that

$$V'(w) = \frac{\partial U(x, w)}{\partial w} = Eu'(w + \tilde{y} - (1 - x)\tilde{l} - \pi x) > 0,$$

where x is the optimal coverage given wealth w . To establish strict concavity, let w_1 and w_2 be distinct levels of wealth and let α_1 and α_2 be strict convex weights; furthermore, let x_i be the optimal coverage for wealth w_i . Then

$$\begin{aligned} V(\alpha_1 w_1 + \alpha_2 w_2) &\geq U(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 w_1 + \alpha_2 w_2) \\ &= Eu(\sum_i \alpha_i (w_i + \tilde{y} - (1 - x_i)\tilde{l} - \pi x_i)) \\ &\geq E \sum_i \alpha_i u(w_i + \tilde{y} - (1 - x_i)\tilde{l} - \pi x_i) \\ &= \sum_i \alpha_i Eu(w_i + \tilde{y} - (1 - x_i)\tilde{l} - \pi x_i) \\ &= \sum_i \alpha_i V(w_i) \end{aligned}$$

Since u is strictly concave, the second inequality will be an exact equality only if

$$w_1 + \tilde{y} - (1 - x_1)\tilde{l} - \pi x_1 = w_2 + \tilde{y} - (1 - x_2)\tilde{l} - \pi x_2$$

with probability 1, or, equivalently, only if

$$(w_1 - w_2) + (\tilde{l} - \pi)(x_1 - x_2) = 0$$

with probability 1. However, since \tilde{l} is not constant with probability 1, this is possible only if $x_1 = x_2$ and $w_1 = w_2$, violating the assumption that w_1 and w_2 are distinct. \square

Basic Two-Period Model without Savings

Let us now extend the basic static Von Neumann-Morgenstern expected utility model to two periods, allowing the agent to purchase insurance in the first period, period 0, to cover potential losses in the following period, period 1. We assume the agent has no means to save, a possibility that is examined in the subsequent subsection.

Consider an agent endowed with predetermined wealth w faces an uncertain income \tilde{y} next period and, additionally, an uncertain, but insurable loss $\tilde{l} \geq 0$

that is independent of income, with $E\tilde{l} > 0$.³ The agent is not permitted to borrow or save, but may insure any portion $x \in [0, 1]$ of the loss at a premium rate $\pi > 0$. That is, if the agent pays a premium $x\pi$ this period, he receives an indemnity $x\tilde{l}$ next period if he experiences a loss of magnitude \tilde{l} . The agent chooses the coverage x that maximizes the sum of current and discounted expected future utility of consumption; that is, he solves:

$$\max_{0 \leq x \leq 1} U(x)$$

where

$$U(x) \equiv u_0(w - \pi x) + \delta E u_1(\tilde{y} - (1 - x)\tilde{l}).$$

Here, $\delta < 1$ is the agent's subjective discount factor and u_0 and u_1 are the agent's current and future utility of consumption, both of which are assumed to be twice continuously differentiable, strictly increasing, and strictly concave. Under these assumptions, U is twice continuously differentiable, strictly increasing, and strictly concave, implying that the optimal coverage x is well-defined and completely characterized by the Karush-Kuhn-Tucker conditions:

$$0 \leq x \leq 1, \quad x > 0 \implies U'(x) \geq 0, \quad x < 1 \implies U'(x) \leq 0.$$

Theorem 5. *The agent purchases coverage if, and only if,*

$$\pi < \pi^* \equiv E\lambda(\tilde{l})\tilde{l}.$$

where

$$\lambda(l) \equiv \delta \frac{E u_1'(\tilde{y} - l)}{u_0'(w)}.$$

Proof. The Karush-Kuhn-Tucker conditions imply that the agent purchases coverage if, and only if,

$$U'(0) = -\pi u_0'(w - \pi x) + \delta E \tilde{l} u_1'(\tilde{y} - (1 - x)\tilde{l}) > 0.$$

The proposition follows immediately, given some algebraic rearrangement of terms. \square

³In order to preclude the possibility of nonpositive wealth in the second period, we further assume that $\underline{y} > \bar{l}$, where $\underline{y} \equiv \sup\{y \mid \Pr(\tilde{y} \leq y) = 0\}$ is the greatest lower bound on attainable income and $\bar{l} \equiv \inf\{l \mid \Pr(\tilde{l} \geq 0) = 1\}$ is the least upper bound on attainable losses.

Theorem 5 states that agent purchases coverage if, and only if, the premium rate π is less than the “risk adjusted” expected loss π^* , defined as the expectation of the loss \tilde{l} weighted by the marginal rate of intertemporal substitution of consumption $\lambda(\tilde{l})$. The risk adjusted expected loss π^* may be greater than or less than the expected loss $E\tilde{l}$, depending on the curvatures of the utility functions and the agent’s wealth w . A sufficiently rich agent (high w) will purchase insurance, even if it is actuarially unfavorable, that is, even if the premium exceeds the expected indemnity. Similarly, a sufficiently poor agent (low w), will not purchase insurance, even if it is actuarially favorable, that is, even if the expected indemnity exceeds the premium.

Theorem 6. *The agent will fully insure, that is, $x = 1$, if, and only if,*

$$\pi \leq \pi^{\text{full}}$$

where π^{full} is uniquely characterized by the nonlinear equation

$$\pi^{\text{full}} \equiv \delta E \frac{u'_1(\tilde{y})}{u'_0(w - \pi^{\text{full}})} \tilde{l}.$$

Moreover, $0 < \pi^{\text{full}} < \pi^*$.

Proof. To prove that π^{full} is well-defined, let

$$f(\pi) \equiv \pi - \delta \frac{Eu'_1(\tilde{y} - \tilde{l})\tilde{l}}{u'_0(w - \pi)}.$$

Now, f is continuous, $f'(\pi) > 0$, $f(0) < 0$, and $\lim_{\pi \rightarrow w} f(\pi) = \infty$, implying that f has a unique positive root on the interval $(0, w)$, which by definition equals π^{full} . Given that U' is strictly decreasing, the Karush-Kuhn-Tucker conditions guarantee that $x \geq 1$ if, and only if,

$$U'(1) = -\pi u'_0(w - \pi) + \delta E \tilde{l} u'_1(\tilde{y} - \tilde{l}) \geq 0$$

or, equivalently, if, and only if,

$$f(\pi) \leq 0 = f(\pi^{\text{full}}),$$

or, equivalently, if, and only if,

$$\pi \leq \pi^{\text{full}}.$$

Also,

$$\pi^{\text{full}} = E \frac{u'_1(\tilde{y})}{u'_0(w - \pi^{\text{full}})} \tilde{l} < E \frac{u'_1(\tilde{y} - \tilde{l})}{u'_0(w)} \tilde{l} = \pi^*$$

since $u'_1(\tilde{y} - \tilde{l}) \geq u'_1(\tilde{y}) > 0$ and $0 < u'_0(w) < u'_0(w - \pi^{\text{full}})$ □

Theorem 6, in conjunction with Theorem 5, states that an agent will fully insure if $\pi \leq \pi^{\text{full}}$, will partially insure if $\pi^{\text{full}} < \pi < \pi^*$, and will not insure if $\pi^* \leq \pi$. If $\pi^{\text{full}} < \pi < \pi^*$, the agent will purchase coverage $x > 0$ that satisfies

$$\pi u'_0(w - \pi x) = \delta E \tilde{l} u'_1(\tilde{y} - (1 - x)\tilde{l}).$$

That is, at optimum coverage, the value of consumption forgone by purchasing a marginal unit of coverage today equals the expected present value of consumption provided by the additional indemnity tomorrow.

Theorem 7. *For $\pi^{\text{full}} < \pi < \pi^*$, the optimal coverage x is a continuously differentiable strictly decreasing function of the premium rate π and a continuously differentiable strictly increasing function of wealth w .*

Proof. For $\pi^{\text{full}} < \pi < \pi^*$, the optimal coverage x is characterized by the nonlinear equation $U'(x) = 0$. The Implicit Function Theorem guarantees that over this range, the optimal coverage is a continuously differentiable function of the premium rate π and wealth w , with partial derivatives

$$\frac{\partial x}{\partial \pi} = \frac{u'_0(w - \pi x) - x \pi u''_0(w - \pi x)}{\pi^2 u''_0(w - \pi x) + \delta E \tilde{l}^2 u''_1(\tilde{y} - (1 - x)\tilde{l})} < 0$$

and

$$\frac{\partial x}{\partial w} = \frac{\pi u''_0(w - \pi x)}{\pi^2 u''_0(w - \pi x) + \delta E \tilde{l}^2 u''_1(\tilde{y} - (1 - x)\tilde{l})} > 0.$$

□

Theorem 8. *Let $V(w)$ denote the maximum sum of current and discounted expected future utility for a given wealth w , that is,*

$$V(w) = \max_{0 \leq x \leq 1} U(x, w)$$

where

$$U(x, w) \equiv u_0(w - \pi x) + \delta E u_1(\tilde{y} - (1 - x)\tilde{l}).$$

Then V is continuously differentiable, strictly increasing, and strictly concave.

Proof. The continuous differentiability and the strict monotonicity of V follow directly from the Envelope Theorem, which guarantees that

$$V'(w) = \frac{\partial U(x, w)}{\partial w} = u'_0(w - \pi x) > 0,$$

where x is the optimal coverage given wealth w . To prove strict concavity, begin by noting that U is jointly strictly concave in (x, w) . Suppose x_i is the optimal coverage for wealth w_i for $i = 1, 2$, where $w_1 \neq w_2$; let $w = \alpha_1 w_1 + \alpha_2 w_2$ and $x = \alpha_1 x_1 + \alpha_2 x_2$ where α_1 and α_2 are strict convex weights. Then,

$$V(w) \geq U(x, w) > \alpha_1 U(x_1, w_1) + \alpha_2 U(x_2, w_2) = \alpha_1 V(w_1) + \alpha_2 V(w_2).$$

□

Suppose now that loss is of the following simple form: $\tilde{l} = L > 0$ with probability p , and equals zero otherwise. The following theorem tells us how the optimal coverage x varies with the magnitude of the loss L , holding both the premium rate π and the expected loss $pL = K$ constant:

Theorem 9. *If $\pi^{\text{full}} < \pi < \pi^*$, then an agent will cover a greater portion of an infrequent catastrophic loss (low p , high L) than a more frequent smaller loss (higher p , lower L) with the same expectation pL and insurable at the same premium rate π .*

Proof. If $\pi^{\text{full}} < \pi < \pi^*$, then the agent will purchase coverage $x < 1$ characterized by $U'(x) = 0$. Applying the Implicit Function Theorem allows us to conclude that

$$\frac{\partial x}{\partial L} = \frac{(1 - x)\delta p L u''_1(\tilde{y} - (1 - x)L)}{\pi^2 u''_0(w - \pi x) + \delta p L^2 u''_1(\tilde{y} - (1 - x)L)} > 0.$$

□

Two-Period Model with Savings

We now extend the two-period model examined in the preceding section to allow for liquid savings.

Consider an agent endowed with predetermined wealth w faces an uncertain income $\tilde{y} \geq 0$ next period and, additionally, an uncertain, but insurable loss $\tilde{l} \geq 0$ that is independent of income, with $E\tilde{l} > 0$.⁴ The agent may save as much of his wealth $s \geq 0$ as he pleases, earning an interest rate $r > 0$. He may also insure any portion $x \geq 0$ of the uncertain loss at a premium rate $\pi > 0$. That is, if the agent pays a premium $x\pi$ this period, he receives an indemnity $x\tilde{l}$ next period if he experiences a loss of magnitude \tilde{l} . The agent chooses the savings s and coverage x that maximize the sum of current and discounted expected future utility of consumption; that is, he solves:

$$\max_{s \geq 0, x \geq 0} U(s, x)$$

where

$$U(s, x) \equiv u_0(w - s - \pi x) + \delta E u_1(\tilde{y} + (1 + r)s - (1 - x)\tilde{l}).$$

Here, $\delta \equiv (1 + \rho)^{-1}$ where $\rho > 0$ is the agent's subjective discount rate and u_0 and u_1 are the agent's current and future utility of consumption, both of which are assumed to be twice continuously differentiable, strictly increasing, and strictly concave.

Let

$$r_s \equiv E\tilde{\lambda}(1 + r) - 1$$

and

$$r_x \equiv E\tilde{\lambda} \frac{\tilde{l}}{\pi} - 1$$

⁴In order to preclude the possibility of nonpositive wealth in the second period, we further assume that $\underline{y} > \bar{l}$, where $\underline{y} \equiv \sup\{y \mid \Pr(\tilde{y} \leq y) = 0\}$ is the greatest lower bound on attainable income and $\bar{l} \equiv \inf\{l \mid \Pr(\tilde{l} \geq 0) = 1\}$ is the least upper bound on attainable losses.

denote, respectively, the expected rates of return on savings and insurance coverage, weighted by the realized marginal rate of intertemporal substitution of consumption

$$\tilde{\lambda} \equiv \frac{u'_1(\tilde{y} + (1+r)s - (1-x)\tilde{l})}{u'_0(w - s - \pi x)}.$$

We refer to r_s and r_x , respectively, as the “risk-adjusted real rates of return” on savings and insurance. Forgoing a marginal unit of consumption today in order to save is expected to yield tomorrow the current equivalent of $1 + r_s$ units of consumption; forgoing a marginal unit of consumption today in order to purchase additional insurance coverage is expected to yield tomorrow the current equivalent of $1 + r_x$ units of consumption tomorrow.

Theorem 10. *The optimal levels of savings s and coverage x satisfy the following conditions:*

$$r_s \leq \rho, \quad s \geq 0, \quad \text{and } r_s < \rho \implies s = 0$$

and

$$r_x \leq \rho, \quad x \geq 0, \quad \text{and } r_x < \rho \implies x = 0$$

Proof. The conclusion is an immediate consequence of the Karush-Kuhn-Tucker Theorem and some algebraic manipulation of the terms. \square

Theorem 10 states that, at an optimum, the risk-adjusted real rates of return on savings and insurance must not exceed the discount rate, for otherwise there would be incentive to save more or purchase more coverage. The agent will not save or purchase insurance coverage if the associated risk-adjusted real rate of return is less than the discount rate. If the agent saves and purchases insurance coverage simultaneously, then the risk-adjusted real rates of return on saving and insurance coverage are equal to the discount rate, and thus to each other.

Let

$$r^* \equiv \frac{u'_0(w)}{\delta E u'_1(\tilde{y} - \tilde{l})} - 1,$$

and

$$\pi^* \equiv \delta \frac{Eu'_1(\tilde{y} - \tilde{l})\tilde{l}}{u'_0(w)}.$$

Theorem 11. *Suppose $r \leq r^*$. Then the agent will not save, and will insure if, and only if, $\pi < \pi^*$.*

Proof. Let s and x be the optimal savings and coverage, given $r < r^*$ and $\pi \geq 0$. If $s > 0$,

$$\begin{aligned} \frac{\partial U}{\partial s} &= -u'_0(w - s - \pi x) + (1 + r)\delta Eu'_1(\tilde{y} + (1 + r)s - (1 - x)\tilde{l}) \\ &< -u'_0(w) + (1 + r)\delta Eu'_1(\tilde{y} - \tilde{l}) \\ &\leq -u'_0(w) + (1 + r^*)\delta Eu'_1(\tilde{y} - \tilde{l}) \\ &= 0 \end{aligned}$$

in violation of the K-K-T conditions. This implies $s = 0$ by contradiction. Given $s = 0$, then for $x = 0$

$$\frac{\partial U}{\partial x} = -\pi u'_0(w) + \delta Eu'_1(\tilde{y} - \tilde{l})\tilde{l} = (\pi^* - \pi)u'_0(w)$$

which implies that

$$\frac{\partial U}{\partial x} \begin{cases} > 0 & \text{if } \pi < \pi^* \\ = 0 & \text{if } \pi = \pi^* \\ < 0 & \text{if } \pi > \pi^*. \end{cases}$$

Thus, the agent will insure if, and only if, $\pi < \pi^*$. □

Theorem 12. *Suppose $\pi \geq \pi^*$. Then the agent will not insure, and will save if, and only if, $r > r^*$.*

Proof. Let s and x be the optimal savings and coverage, given $\pi \geq \pi^*$ and

r . If $x > 0$,

$$\begin{aligned}
\frac{\partial U}{\partial x} &= -\pi u'_0(w - s - \pi x) + \delta E u'_1(\tilde{y} + (1+r)s - (1-x)\tilde{l})\tilde{l} \\
&< -\pi u'_0(w) + (1+r)\delta E u'_1(\tilde{y} - \tilde{l})\tilde{l} \\
&\leq -\pi^* u'_0(w) + (1+r)\delta E u'_1(\tilde{y} - \tilde{l})\tilde{l} \\
&= 0
\end{aligned}$$

in violation of the K-K-T conditions. This implies $x = 0$ by contradiction. Given $x = 0$, then for $s = 0$

$$\frac{\partial U}{\partial s} = -u'_0(w) + (1+r)\delta E u'_1(\tilde{y} - \tilde{l})$$

which implies that

$$\frac{\partial U}{\partial s} \begin{cases} > 0 & \text{if } r > r^* \\ = 0 & \text{if } r = r^* \\ < 0 & \text{if } r < r^*. \end{cases}$$

Thus, the agent will save a positive amount if, and only if, $r > r^*$. \square

Theorem 13. *The critical interest rate r^* below which an agent will not save is a strictly decreasing function of wealth w ; the critical premium π^* above which an agent will not insure is a strictly increasing function of wealth w .*

Proof. Differentiating expressions above

$$\frac{\partial r^*}{\partial w} = \frac{u''_0(w)}{\delta E u'_1(\tilde{y} - \tilde{l})} < 0.$$

and

$$\frac{\partial \pi^*}{\partial w} = -\delta \frac{u''_0(w) E u'_1(\tilde{y} - \tilde{l}) \tilde{l}}{(u'_0(w))^2} > 0.$$

\square

Thus, given fixed interest rate r and premium rate π , the wealthier the agent, the more likely he is to save and insure.

Theorem 14. *If $r > r^*$, then there exists π_1^* and π_2^* , where $0 < \pi_1^* < \pi_2^* < \pi^*$, such that*

- *The agent insures, but does not save if $\pi \leq \pi_1^*$.*
- *The agent saves, but does not insure if $\pi \geq \pi_2^*$.*
- *The agent both saves and insures if $\pi_1^* < \pi < \pi_2^*$.*

Proof. The critical premium level π_1^* is characterized jointly with the optimal coverage x^* at that premium by the pair of nonlinear equations

$$u'_0(w - \pi_1^* x^*) = (1 + r) \delta E u'_1(\tilde{y} - (1 - x^*) \tilde{l})$$

$$\pi_1^* u'_0(w - \pi_1^* x^*) = \delta E u'_1(\tilde{y} - (1 - x^*) \tilde{l}) \tilde{l}.$$

The critical premium level π_2^* is characterized jointly with the optimal savings s^* at that and higher premiums by the pair of nonlinear equations

$$u'_0(w - s^*) = (1 + r) \delta E u'_1(\tilde{y} + (1 + r) s^* - \tilde{l})$$

$$\pi_2^* u'_0(w - s^*) = \delta E u'_1(\tilde{y} + (1 + r) s^* - \tilde{l}) \tilde{l}.$$

□

Figure 1 illustrates how optimal savings and insurance choices divide the $r - \pi$ plane into four distinct sections. For low interest rates, below r^* , and high premium rates, above π^* , it is optimal to neither save nor insure. In the northeast section, where interest and premium rates are relatively high, it is optimal to save, but not to insure. In the southwest section, where interest and premium rates are relatively low, it is optimal to insure, but not to save. It is optimal to both save and insure only for interest-premium rate combinations that lie between π_1^* and π_2^* .

Figure 2 illustrates the demand for savings with access to insurance (blue) and without access to insurance (red). The demand for savings is not affected by access to insurance for low interest rates, $r \leq r^*$, because these interest

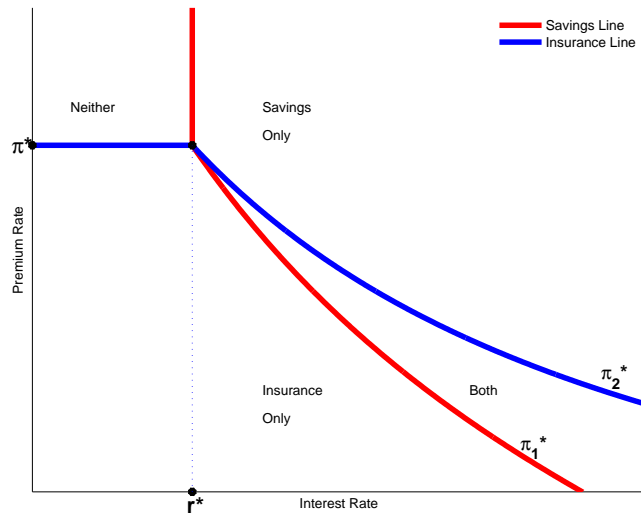


Figure 1: Savings and Insurance Choices, Interest-Premium Rate Plane

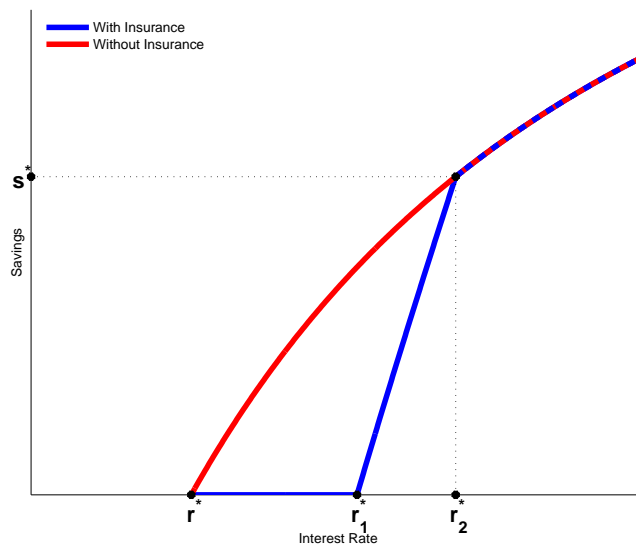


Figure 2: Demand for Savings, Fixed Premium Rate

rates are too low to merit savings, regardless of whether insurance is available. The demand for savings is not affected by access to insurance for high interest rates, $r > r_2^*$, because at high interest rates it is more beneficial to save than to insure, and the agent will not insure. For intermediate interest rates, $r^* < r < r_2^*$, insurance “crowds out” savings, reducing it below what it would be without insurance.

Theorem 15. *If $\pi < \pi^*$, then there exists r_1^* and r_2^* , where $r^* < r_1^* < r_2^*$, such that*

- *The agent insures, but does not save if $r \leq r_1^*$.*
- *The agent saves, but does not insure if $r \geq r_2^*$.*
- *The agent both saves and insures if $r_1^* < r < r_2^*$.*

Proof. The critical interest rate r_1^* is characterized jointly with the optimal coverage x^* at that and lower interests rate by the pair of nonlinear equations

$$\begin{aligned} u'_0(w - \pi x^*) &= (1 + r_1^*)\delta E u'_1(\tilde{y} - (1 - x^*)\tilde{l}) \\ \pi u'_0(w - \pi x^*) &= \delta E u'_1(\tilde{y} - (1 - x^*)\tilde{l})\tilde{l}. \end{aligned}$$

The critical interest rate r_2^* is characterized jointly with the optimal savings s^* at that interest rate by the pair of nonlinear equations

$$\begin{aligned} u'_0(w - s^*) &= (1 + r_2^*)\delta E u'_1(\tilde{y} + (1 + r_2^*)s^* - \tilde{l}) \\ \pi u'_0(w - s^*) &= \delta E u'_1(\tilde{y} + (1 + r_2^*)s^* - \tilde{l})\tilde{l}. \end{aligned}$$

□

Figure 3 illustrates how optimal savings and insurance choices divide the $\pi - r$ plane into four distinct sections. For low interest rates below r^* and high premium rates above π^* , it optimal to neither save nor insure. In the northeast section, where interest and premium rates are relatively high, it is optimal to save, but not to insure. In the southwest section, where interest and premium rates are relatively low, it is optimal to insure, but not to save. It is optimal to both save and insure only for premium-interest rate combinations that lie between the r_1^* and r_2^* curves.

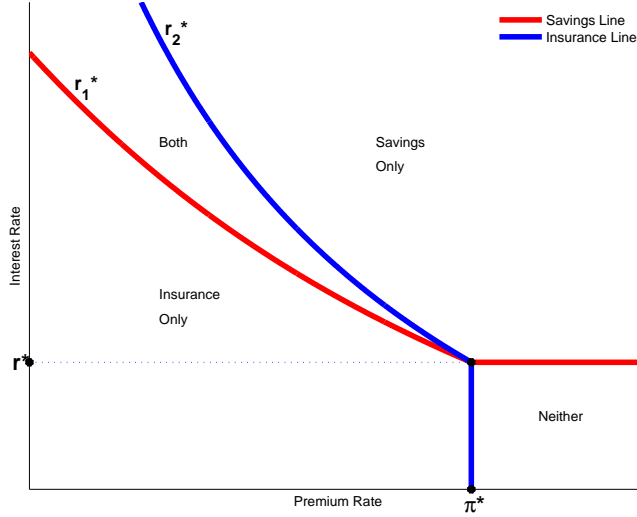


Figure 3: Savings and Insurance Choices, Premium-Interest Rate Plane

Figure 4 illustrates the demand for insurance with access to savings (blue) and without access to savings (red). The demand for insurance is not affected by access to savings for high premium rates, $\pi \geq \pi^*$, because these premium rates are too high to merit purchasing coverage, regardless of whether one has access to savings. The demand for insurance is not affected by access to savings for low premium rates, $\pi < \pi_1^*$, because at low premium rates it is more beneficial to insure than to save, and no savings is undertaken. For intermediate premium rates, $\pi_1^* < \pi < \pi^*$, savings “crowds out” insurance, reducing it below what it would be without access to savings.

Theorem 16. *Let $V(w)$ denote the maximum sum of current and discounted expected future utility for a given wealth w , that is,*

$$V(w) = \max_{s \geq 0, x \geq 0} U(s, x, w)$$

where

$$U(s, x, w) \equiv u_0(w - s - \pi x) + \delta E u_1(\tilde{y} + (1 + r)s - (1 - x)\tilde{l}).$$

Then V is continuously differentiable, strictly increasing, and strictly concave.

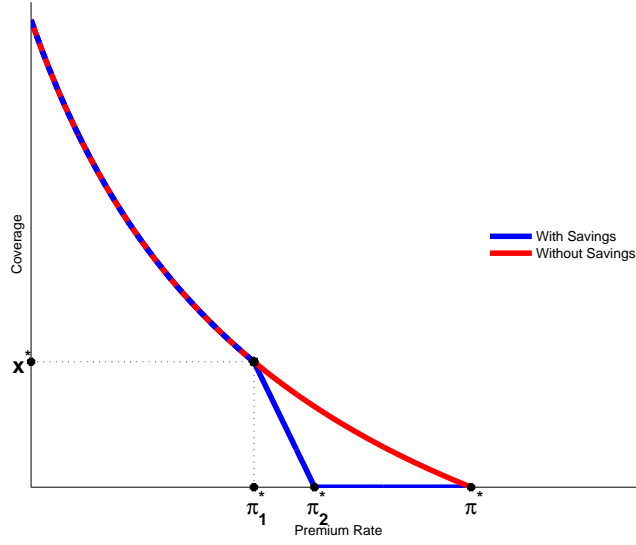


Figure 4: Demand for Insurance, Fixed Interest Rate

Proof. The continuous differentiability and the strict monotonicity of V follow directly from the Envelope Theorem, which guarantees that

$$V'(w) = \frac{\partial U(s, x, w)}{\partial w} = u'_0(w - s - \pi x) > 0,$$

where s and x are, respectively, the optimal savings and coverage given wealth w . To prove strict concavity, begin by noting that U is jointly strictly concave in (s, x, w) . Suppose s_i and x_i are, respectively, the optimal savings and coverage for wealth w_i for $i = 1, 2$, where $w_1 \neq w_2$; further suppose that $w = \alpha_1 w_1 + \alpha_2 w_2$, $s = \alpha_1 s_1 + \alpha_2 s_2$, and $x = \alpha_1 x_1 + \alpha_2 x_2$ where α_1 and α_2 are strict convex weights. Then,

$$V(w) \geq U(s, x, w) > \alpha_1 U(s_1, x_1, w_1) + \alpha_2 U(s_2, x_2, w_2) = \alpha_1 V(w_1) + \alpha_2 V(w_2).$$

□

Two-Period Model with Saving - Wealth Effects

For this analysis, we assume that the interest rate $r > 0$ and the premium rate $\pi > 0$ are both given and examine how savings and insurance decisions vary with wealth.

Let w_r^* and w_π^* be implicitly defined by

$$r \equiv \frac{u'_0(w_r^*)}{\delta E u'_1(\tilde{y} - \tilde{l})} - 1,$$

and

$$\pi \equiv \delta \frac{E u'_1(\tilde{y} - \tilde{l}) \tilde{l}}{u'_0(w_\pi^*)}.$$

Equivalently, let

$$w_r^* \equiv u_0'^{-1} \left((1 + r) \delta E u'_1(\tilde{y} - \tilde{l}) \right),$$

and

$$w_\pi^* \equiv u_0'^{-1} \left(\delta \frac{E u'_1(\tilde{y} - \tilde{l}) \tilde{l}}{\pi} \right).$$

Clearly, due to the curvature properties assumed for the utility function u , both are well defined, provided we assume that $\lim_{w \rightarrow 0} u'_0(w) = \infty$ and $\lim_{w \rightarrow \infty} u'_0(w) = 0$, which we will assume. Moreover:

Theorem 17. *Suppose $w \leq w_r^*$. Then the agent will not save, and will insure if, and only if, $w > w_\pi^*$.*

Proof. Follows from Theorem 26. □

Theorem 18. *Suppose $w \leq w_\pi^*$. Then the agent will not insure, and will save if, and only if, $w > w_r^*$.*

Proof. Follows from Theorem 27. □

Theorem 19. *The critical wealth w_r^* below which an agent will not save is a strictly decreasing function of the interest rate r ; the critical wealth w_π^* below which an agent will not insure is a strictly increasing function of the premium rate π .*

Proof. Follows from Theorem 28. □

Let

$$\bar{\pi} \equiv \frac{1}{1+r} \frac{Eu'_1(\tilde{y} - \tilde{l})\tilde{l}}{Eu'_1(\tilde{y} - \tilde{l})}$$

denote the premium rate at which the agent will be indifferent to purchasing insurance in the absence of savings. Then we have two theorems:

Theorem 20.

$$\pi \begin{cases} > \\ = \\ < \end{cases} \bar{\pi} \iff w_\pi^* \begin{cases} > \\ = \\ < \end{cases} w_r^*$$

Proof. Follows from Theorem 28. □

Theorem 21. *Optimal coverage is a strictly increasing function of wealth over the interval in which the agent insures but does not save.*

Proof. Over the stated interval,

$$-\pi u'_0(w - \pi x) + \delta Eu'_1(\tilde{y} - (1-x)\tilde{l})\tilde{l} = 0.$$

Differentiating latter expression and rearranging:

$$x'(w) = \frac{\pi u''_0(w - \pi x)}{\delta Eu''_1(\tilde{y} - (1-x)\tilde{l})\tilde{l}^2 + \pi^2 u''_0(w - \pi x)} > 0$$

□

Further theoretical results of interest are possible only if we place additional restrictions on the model. The critical assumption needed to derive these results appears to be the following:

Assumption 1. *Regularity Assumption:* For $\gamma \geq 0$,

$$\bar{\pi}(\gamma) \equiv \frac{1}{1+r} \frac{Eu'_1(\tilde{y} + \gamma - \tilde{l})\tilde{l}}{Eu'_1(\tilde{y} + \gamma - \tilde{l})}$$

is a non-increasing function of γ .

The Regularity assumption states that the maximum premium rate the agent is willing to pay for insurance will not increase if income experiences a spread-preserving increase in its mean. More basic assumptions that guarantee Regularity are not easy to come by.

Theorem 22. *Suppose the Regularity Assumption holds. Then, if, $\pi > \bar{\pi}$, the agent does not insure at any level of wealth.*

Proof. To prove the contrapositive, suppose the agent insures for some level of wealth. Then there is a $w \geq w_\pi^*$, such that $s \geq 0$ and $x = 0$ are optimal for w and

$$\begin{aligned} -u'_0(w - s) + \delta(1+r)Eu'_1(\tilde{y} + (1+r)s - \tilde{l}) &\leq 0 \\ -\pi u'_0(w - s) + \delta Eu'_1(\tilde{y} + (1+r)s - \tilde{l})\tilde{l} &= 0, \end{aligned}$$

so that

$$\bar{\pi} \geq \bar{\pi}((1+r)s) = \frac{1}{1+r} \frac{Eu'_1(\tilde{y} + (1+r)s - \tilde{l})\tilde{l}}{Eu'_1(\tilde{y} + (1+r)s - \tilde{l})} \geq \pi.$$

□

Theorem 23. *Assume that*

- *the agent's period 1 utility function u_1 exhibits non-increasing absolute risk aversion*

$$A(w) \equiv -\frac{u''_1(w)}{u'_1(w)}$$

at all levels of wealth $w > 0$;

- the loss \tilde{l} is binary, equaling $L > 0$ with probability p and 0 with probability $1 - p$;
- income \tilde{y} is deterministic, and equals y with probability 1.

Then the Regularity Assumption holds.

Proof. Note that, under the given assumptions,

$$\frac{d\bar{\pi}}{d\gamma} = \frac{p(1-p)}{1+r} \frac{u'_1(y+\gamma)u'_1(y+\gamma-L)}{Eu'_1(\tilde{y}+\gamma-\tilde{l})} (A(y+\gamma) - A(y+\gamma-L)) \leq 0.$$

□

Note 2. Recall that constant absolute risk aversion and constant relative risk aversion utility functions satisfy the DARA assumption of the preceding theorem.

What can we say about the proportion of wealth invested in insurance, $\pi x(w)/w$? The derivative of this proportion has the same sign as

$$wx'(w) - x(w),$$

which is positive if and only if

$$\frac{\pi^2 u''_0(w - \pi x)}{\delta Eu''_1(\tilde{y} - (1-x)\tilde{l})\tilde{l}^2 + \pi^2 u''_0(w - \pi x)} > x(w)/w.$$

However, this is hard to sign.

What can we say about the signs of $x'(w)$ and $s'(w)$ when $s > 0$ and $x > 0$?

Here,

$$0 = -u'_0(w - s - \pi x) + \delta(1+r)Eu'_1(\tilde{y} + (1+r)s - (1-x)\tilde{l}).$$

$$0 = -\pi u'_0(w - s - \pi x) + \delta Eu'_1(\tilde{y} + (1+r)s - (1-x)\tilde{l}).$$

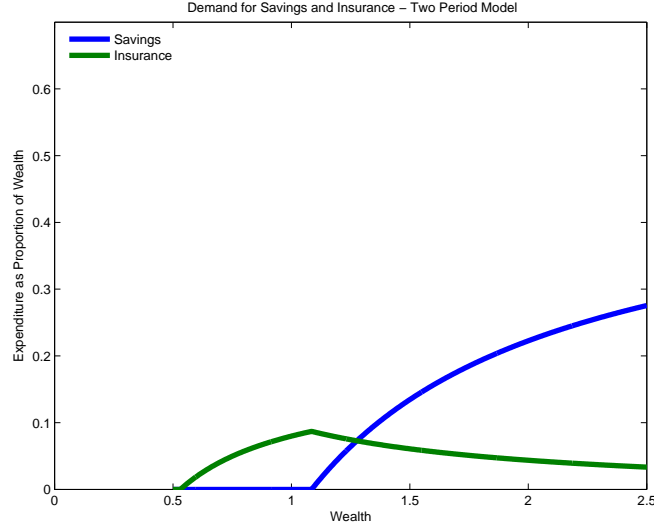


Figure 5: Demand for Savings and Insurance as a Function of Wealth

Totally differentiating

$$0 = 1 + (R^2 Eb(\tilde{l}) - 1)s' + (REb(\tilde{l})\tilde{l} - \pi)x'.$$

$$0 = 1 + (REb(\tilde{l})\frac{\tilde{l}}{\pi} - 1)s' + (Eb(\tilde{l})\frac{\tilde{l}^2}{\pi} - \pi)x'.$$

where

$$b(\tilde{l}) = -\delta \frac{u_1''(\tilde{y} + (1+r)s - (1-x)\tilde{l})}{u_0''(w - s - \pi x)} < 0$$

But this is nearly impossible to sign.

Basic Results

$$\frac{\partial U}{\partial s} = -u_0'(w - s - \pi x) + \delta(1+r)Eu_1'(\tilde{y} + (1+r)s - (1-x)\tilde{l}).$$

$$\frac{\partial U}{\partial x} = -\pi u_0'(w - s - \pi x) + \delta Eu_1'(\tilde{y} + (1+r)s - (1-x)\tilde{l})\tilde{l}.$$

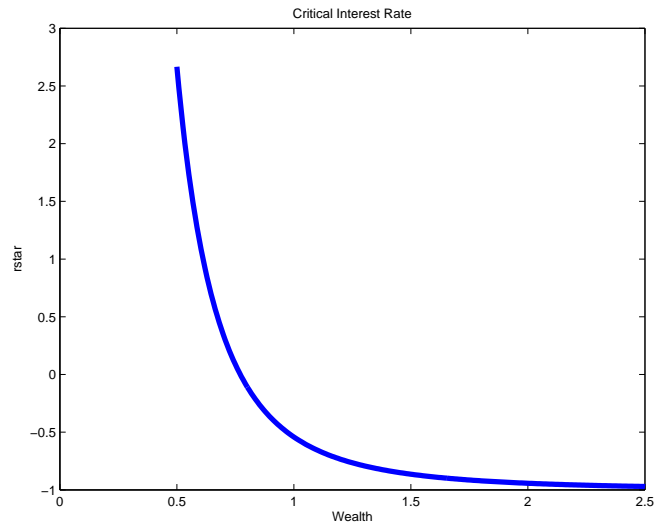


Figure 6: Critical Interest Rate

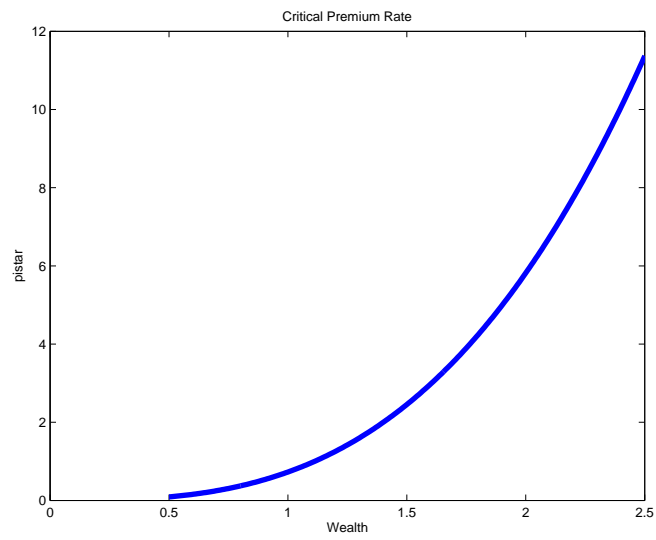


Figure 7: Critical Premium Rate

FIRST CRITICAL POINT

$$0 = -\pi u'_0(w) + \delta E u'_1(\tilde{y} - \tilde{l})\tilde{l}.$$

provided

$$-u'_0(w) + \delta(1+r)E u'_1(\tilde{y} - \tilde{l}) \leq 0$$

which would imply

$$\pi(1+r)E u'_1(\tilde{y} - \tilde{l}) \leq E u'_1(\tilde{y} - \tilde{l})\tilde{l}$$

SECOND CRITICAL POINT

$$0 = -\pi u'_0(w - \pi x) + \pi \delta(1+r)E u'_1(\tilde{y} - (1-x)\tilde{l}).$$

$$0 = -\pi u'_0(w - \pi x) + \delta E u'_1(\tilde{y} - (1-x)\tilde{l})\tilde{l}.$$

THIRD CRITICAL POINT

$$0 = -u'_0(w - s) + \delta(1+r)E u'_1(\tilde{y} + (1+r)s - \tilde{l}).$$

$$0 = -\pi u'_0(w - s) + \delta E u'_1(\tilde{y} + (1+r)s - \tilde{l})\tilde{l}.$$

so that

$$\pi(1+r)Eu'_1(\tilde{y} + (1+r)s - \tilde{l}) = Eu'_1(\tilde{y} + (1+r)s - \tilde{l})\tilde{l}.$$

for current parameterization, let

$$u_L = u'(\tilde{y} + (1+r)s - L)$$

and

$$u_0 = u'(\tilde{y} + (1+r)s)$$

$$\pi(1+r)[pu_L + (1-p)u_0] = pLu_L$$

=====

Infinite-Horizon Model with Savings

An infinitely-lived agent begins each period endowed with predetermined wealth w , which he must then allocate among consumption, savings, and purchases of insurance. The agent faces an uncertain income $\tilde{y} \geq 0$ the following period and, additionally, an uncertain, but insurable loss $\tilde{l} \geq 0$ that is independent of income, with $E\tilde{l} > 0$.⁵ The agent may save as much of his wealth $s \geq 0$ as he pleases, earning a per-period interest rate $r > 0$. He may also insure any portion $x \geq 0$ of the uncertain loss the following period at a premium rate $\pi > 0$. That is, if the agent pays a premium $x\pi$

⁵In order to preclude the possibility of nonpositive wealth in any period, we further assume that $\underline{y} > \bar{l}$, where $\underline{y} \equiv \sup\{y | \Pr(\tilde{y} \leq y) = 0\}$ is the greatest lower bound on attainable income and $\bar{l} \equiv \inf\{l | \Pr(\tilde{l} \geq 0) = 1\}$ is the least upper bound on attainable losses.

this period, he receives an indemnity $x\tilde{l}$ next period if he experiences a loss of magnitude \tilde{l} .

The agent chooses the savings s and coverage x that maximize the sum of current and discounted expected future utility of consumption over an infinite horizon. By Bellman's Principle of Optimality, the agent's value function, $V(w)$, which denotes the maximum attainable sum of current and discounted expected future utility of consumption given the agent's current wealth w , is characterized by the functional equation

$$V(w) = \max_{s \geq 0, x \geq 0} \{u(w - s - \pi x) + \delta EV(\tilde{y} + (1 + r)s - (1 - x)\tilde{l})\}.$$

Here, $\delta \equiv (1 + \rho)^{-1}$ where $\rho > 0$ is the agent's subjective discount rate and u is the agent's utility of consumption, which is presumed to be twice continuously differentiable, strictly increasing, and strictly concave. The agent's consumption equals his predetermined wealth, plus borrowing, less production costs, less index insurance premium payments.

Theorem 24. *The agent's Bellman functional equation possesses an unique solution V , which denotes the maximum sum of current and discounted expected future utility for a given wealth w . The value function V is continuous, strictly increasing, and strictly concave.*

Proof. Need special arguments. Not sure about differentiability. See Stokey-Lucas. \square

We denote by $s(w)$ and $x(w)$ the optimal levels of savings and insurance coverage, respectively, given wealth w . These are the agent's optimal policy functions.

For any admissible level of wealth w , let

$$r_s(w) \equiv E\tilde{\lambda}(w; \tilde{y}, \tilde{l})(1 + r) - 1$$

and

$$r_x(w) \equiv E\tilde{\lambda}(w; \tilde{y}, \tilde{l})\frac{\tilde{l}}{\pi} - 1$$

denote, respectively, the expected rates of return on savings and insurance coverage, weighted by the realized marginal rate of intertemporal substitution of consumption

$$\tilde{\lambda}(w; \tilde{y}, \tilde{l}) \equiv \frac{V'(\tilde{y} + (1+r)s(w) - (1-x(w))\tilde{l})}{u'(w - s(w) - \pi x(w))}.$$

We refer to $r_s(w)$ and $r_x(w)$, respectively, as the “risk-adjusted real rates of return” on savings and insurance. Forgoing a marginal unit of consumption today in order to save is expected to yield tomorrow the current equivalent of $1 + r_s$ units of consumption; forgoing a marginal unit of consumption today in order to purchase additional insurance coverage is expected to yield tomorrow the current equivalent of $1 + r_x$ units of consumption tomorrow.

Theorem 25. *For any given level of wealth w , the optimal levels of savings $s(w)$ and coverage $x(w)$ satisfy the following conditions:*

$$r_s(w) \leq \rho, \quad s(w) \geq 0, \quad \text{and } r_s(w) < \rho \implies s(w) = 0$$

and

$$r_x(w) \leq \rho, \quad x(w) \geq 0, \quad \text{and } r_x(w) < \rho \implies x(w) = 0$$

Proof. The conclusion is an immediate consequence of the Karush-Kuhn-Tucker Theorem and some algebraic manipulation of the terms. \square

Theorem 25 states that, at an optimum, the risk-adjusted real rates of return on savings and insurance must not exceed the discount rate, for otherwise there would be incentive to save more or purchase more coverage. The agent will not save or purchase insurance coverage if the associated risk-adjusted real rate of return is less than the discount rate. If the agent saves and purchases insurance coverage simultaneously, then the risk-adjusted real rates of return on saving and insurance coverage are equal to the discount rate, and thus to each other.

For any given level of wealth w , let

$$r^*(w) \equiv \frac{u'(w)}{\delta EV'(\tilde{y} - \tilde{l})} - 1,$$

and

$$\pi^*(w) \equiv \delta \frac{EV'(\tilde{y} - \tilde{l})\tilde{l}}{u'(w)}.$$

Theorem 26. *Suppose $r \leq r^*(w)$. Then the agent will not save, and will insure if, and only if, $\pi < \pi^*(w)$.*

Proof. Let s and x be the optimal savings and coverage, given level of wealth w , and suppose $r < r^*(w)$. If $s > 0$,

$$\begin{aligned} \frac{\partial U}{\partial s} &= -u'(w - s - \pi x) + (1 + r)\delta EV'(\tilde{y} + (1 + r)s - (1 - x)\tilde{l}) \\ &< -u'(w) + (1 + r)\delta EV'(\tilde{y} - \tilde{l}) \\ &\leq -u'(w) + (1 + r^*)\delta EV'(\tilde{y} - \tilde{l}) \\ &= 0 \end{aligned}$$

in violation of the K-K-T conditions. This implies $s = 0$ by contradiction. Given $s = 0$, then for $x = 0$

$$\frac{\partial U}{\partial x} = -\pi u'(w) + \delta EV'(\tilde{y} - \tilde{l})\tilde{l} = (\pi^* - \pi)u'(w)$$

which implies that

$$\frac{\partial U}{\partial x} \begin{cases} > 0 & \text{if } \pi < \pi^* \\ = 0 & \text{if } \pi = \pi^* \\ < 0 & \text{if } \pi > \pi^*. \end{cases}$$

Thus, the agent will insure if, and only if, $\pi < \pi^*$. □

Theorem 27. *Suppose $\pi \geq \pi^*(w)$. Then the agent will not insure, and will save if, and only if, $r > r^*(w)$.*

Proof. Let s and x be the optimal savings and coverage, given $\pi \geq \pi^*$ and

r . If $x > 0$,

$$\begin{aligned}
\frac{\partial U}{\partial x} &= -\pi u'(w - s - \pi x) + \delta EV'(\tilde{y} + (1 + r)s - (1 - x)\tilde{l})\tilde{l} \\
&< -\pi u'(w) + (1 + r)\delta EV'(\tilde{y} - \tilde{l})\tilde{l} \\
&\leq -\pi^* u'(w) + (1 + r)\delta EV'(\tilde{y} - \tilde{l})\tilde{l} \\
&= 0
\end{aligned}$$

in violation of the K-K-T conditions. This implies $x = 0$ by contradiction. Given $x = 0$, then for $s = 0$

$$\frac{\partial U}{\partial s} = -u'(w) + (1 + r)\delta EV'(\tilde{y} - \tilde{l})$$

which implies that

$$\frac{\partial U}{\partial s} \begin{cases} > 0 & \text{if } r > r^* \\ = 0 & \text{if } r = r^* \\ < 0 & \text{if } r < r^*. \end{cases}$$

Thus, the agent will save a positive amount if, and only if, $r > r^*$. \square

Theorem 28. *The critical interest rate $r^*(w)$ below which an agent will not save is a strictly decreasing function of wealth w ; the critical premium $\pi^*(w)$ above which an agent will not insure is a strictly increasing function of wealth w .*

Proof. Differentiating expressions above

$$\frac{\partial r^*(w)}{\partial w} = \frac{u''(w)}{\delta EV'(\tilde{y} - \tilde{l})} < 0.$$

and

$$\frac{\partial \pi^*(w)}{\partial w} = -\delta \frac{u''(w)EV'_1(\tilde{y} - \tilde{l})\tilde{l}}{(u'(w))^2} > 0.$$

\square

Thus, the wealthier the agent, the higher must be the interest rate to induce him to save and the lower must be the premium rate to induce him to insure.

Let w_r^* and w_π^* be defined by

$$w_r^* = u'^{-1} \left((1+r)\delta EV'(\tilde{y} - \tilde{l}) \right),$$

and

$$w_\pi^* \equiv u'^{-1} \left(\frac{\delta EV'(\tilde{y} - \tilde{l})\tilde{l}}{\pi} \right).$$

so that $r^*(w_r^*) = r$ and $\pi^*(w_\pi^*) = \pi$. Clearly, due to the curvature properties assumed for the utility function u , both are well defined, provided we assume that $\lim_{w \rightarrow 0} u'(w) = \infty$ and $\lim_{w \rightarrow \infty} u'(w) = 0$, which we will assume. Moreover:

Theorem 29. *Suppose $w \leq w_r^*$. Then the agent will not save, and will insure if, and only if, $w > w_\pi^*$.*

Proof. Follows from Theorem 26. □

Theorem 30. *Suppose $w \leq w_\pi^*$. Then the agent will not insure, and will save if, and only if, $w > w_r^*$.*

Proof. Follows from Theorem 27. □

Theorem 31. *The critical wealth w_r^* below which an agent will not save is a strictly decreasing function of the interest rate r ; the critical wealth w_π^* below which an agent will not insure is a strictly increasing function of the premium rate π .*

Proof. I am not sure that this is easy to prove, and may actually be false. See Theorem 28. □

Let

$$\bar{\pi} \equiv \frac{1}{1+r} \frac{EV'(\tilde{y} - \tilde{l})\tilde{l}}{EV'(\tilde{y} - \tilde{l})}$$

denote the premium rate at which the agent will be indifferent to purchasing insurance in the absence of savings. Then we have two theorems:

Theorem 32.

$$\pi \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \bar{\pi} \iff w_{\pi}^* \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} w_r^*$$

Proof. Follows from Theorem 28. Double check this. □

Theorem 33. *Optimal coverage is a strictly increasing function of wealth over the interval in which the agent insures but does not save.*

Proof. Over the stated interval,

$$-\pi u'(w - \pi x) + \delta EV'(\tilde{y} - (1 - x)\tilde{l})\tilde{l} = 0.$$

Differentiating latter expression and rearranging:

$$x'(w) = \frac{\pi u''(w - \pi x)}{\delta EV''(\tilde{y} - (1 - x)\tilde{l})\tilde{l}^2 + \pi^2 V''(w - \pi x)} > 0$$

□

Further theoretical results of interest are possible only if we place additional restrictions on the model. The critical assumption needed to derive these results appears to be the following:

Assumption 2. *Regularity Assumption: For $\gamma \geq 0$,*

$$\bar{\pi}(\gamma) \equiv \frac{1}{1+r} \frac{EV'(\tilde{y} + \gamma - \tilde{l})\tilde{l}}{EV'(\tilde{y} + \gamma - \tilde{l})}$$

is a non-increasing function of γ .

The Regularity assumption states that the maximum premium rate the agent is willing to pay for insurance will not increase if income experiences a spread-preserving increase in its mean. More basic assumptions that guarantee Regularity are not easy to come by.

Theorem 34. *Suppose the Regularity Assumption holds. Then, if, $\pi > \bar{\pi}$, the agent does not insure at any level of wealth.*

Proof. To prove the contrapositive, suppose the agent insures for some level of wealth. Then there is a $w \geq w_{\pi}^*$, such that $s \geq 0$ and $x = 0$ are optimal for w and

$$\begin{aligned} -u'(w - s) + \delta(1 + r)EV'(\tilde{y} + (1 + r)s - \tilde{l}) &\leq 0 \\ -\pi u'(w - s) + \delta EV'(\tilde{y} + (1 + r)s - \tilde{l})\tilde{l} &= 0, \end{aligned}$$

so that

$$\bar{\pi} \geq \pi((1 + r)s) = \frac{1}{1 + r} \frac{EV'(\tilde{y} + (1 + r)s - \tilde{l})\tilde{l}}{EV'(\tilde{y} + (1 + r)s - \tilde{l})} \geq \pi.$$

□

Theorem 35. *Assume that*

- *the agent's value function V exhibits non-increasing absolute risk aversion*

$$A(w) \equiv -\frac{V''(w)}{V'(w)}$$

at all levels of wealth $w > 0$;

- *the loss \tilde{l} is binary, equaling $L > 0$ with probability p and 0 with probability $1 - p$;*
- *income \tilde{y} is deterministic, and equals y with probability 1.*

Then the Regularity Assumption holds.

Proof. Note that, under the given assumptions,

$$\frac{d\bar{\pi}}{d\gamma} = \frac{p(1 - p)}{1 + r} \frac{V'(y + \gamma)V'(y + \gamma - L)}{Eu'_1(\tilde{y} + \gamma - \tilde{l})} (A(y + \gamma) - A(y + \gamma - L)) \leq 0.$$

□

References

- Carroll, Christopher D. 1997. "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis." *Quarterly Journal of Economics* 112(1):1–55.
- Deaton, Angus. 1991. "Saving and Liquidity Constraints." *Econometrica* 59(5):1221–1248.
- . 1992. "Household Saving in LDCs: Credit Markets, Insurance and Welfare." *Scandinavian Journal of Economics* 94(2):253–273.
- Dercon, Stefan. 2002. "Income Risk, Coping Strategies, and Safety Nets." *World Bank Research Observer* 17(2):141–166.
- Ehrlich, Isaac and Gary S. Becker. 1972. "Market Insurance, Self-Insurance, and Self-Protection." *Journal of Political Economy* 80(4):623–648.
- Eisenhauer, Joseph E. 1994. "Household Use of Open-End Credit to Finance Risk." *Journal of Consumer Affairs* 28(1):154–169.
- Gollier, Christian. 1994. "Insurance and Precautionary Capital Accumulation in a Continuous-Time Model." *Journal of Risk and Insurance* 61(1):78–95.
- . 2001. *The Economics of Risk and Time*. Cambridge, MA: MIT Press.
- . 2002. "Time Diversification, Liquidity Constraints, and Decreasing Aversion to Risk on Wealth." *Journal of Monetary Economics* 49(7):1439–1459.
- . 2003. "To Insure or Not to Insure? An Insurance Puzzle." *Geneva Papers on Risk and Insurance - Theory* 28(1):5–24.
- Heaton, John and Deborah J. Lucas. 1996. "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing." *Journal of Political Economy* 104(13):443–487.
- Hubbard, Glenn, Jonathan Skinner, and Stephen P. Zeldes. 1995. "Precautionary Saving and Social Insurance." *Journal of Political Economy* 103(2):360–399.

- Knight, Thomas O. and Keith H. Coble. 1997. "Survey of U.S. Multiple Peril Crop Insurance Literature since 1980." *Review of Agricultural Economics* 19(1):128–156.
- Kurosaki, Takashi and Marcel Fafchamps. 2002. "Insurance Market Efficiency and Crop Choices in Pakistan." *Journal of Development Economics* 67(2):4–453.
- Morduch, Jonathan. 1995. "Income Smoothing and Consumption Smoothing." *Journal of Economic Perspectives* 9(3):103–114.
- Paxson, Christina H. 1992. "Using Weather Variability to Estimate the Response of Savings to Transitory Income in Thailand." *American Economic Review* 82(1):15–33.