- **Public Goods**

  **- public goods can be consumed collectively by all households - violates assumption of private goods in the Arrow-Debreu economy**

  **- formal analysis of efficient provision of public goods begun by Samuelson (1954)**

- **Definitions:**

  **(i) Pure Public Goods**

  **A pure public good is one that can accommodate any number of users, and it has the following properties:**

  **-** ***non-excludability*** **- if a public good is supplied, no household can be excluded from consuming it, except at infinite cost**

  **-** ***non-rivalry*** **- consumption of the public good by one household does not reduce quantity available for consumption by other households**

**- implication of non-excludability is that consumption cannot be controlled efficiently by the price system**

**- non-rivalry implies that all households can simultaneously consume a quantity of the public good equal to its total supply**

**- if consumption of the public good can be reduced at no cost, it satisfies concept of *free disposal***

**(ii) Impure Public Goods**

**- in practice, public goods tend to suffer from congestion, i.e., congestion results in a reduction in the return from using a public good as use of a given supply increases**

## ■ Optimal Provision

**(i) Pure Public Good**

**- following Samuelson, assume a single public good, with no disposal, i.e., each household consumes an amount of the public good equal to its supply**

**- there are *H* households, *h=1,...,H*, each with a utility function:**

$$U^h = U^h(x^h, G) \qquad (1)$$

**$x^h$ is consumption by household *h* of a vector of private goods, *G* is supply of public good, appearing in all utility functions**

**- feasible production set is:**

$$F(X, G) \leq 0 \qquad (2)$$

**where:**

$$X = \sum_{h=1}^{H} x^h$$

**- to establish**

**Pareto efficient allocation, government chooses $x^h$ and *G* to maximize utility of *h*=1, subject to *h*=2 to *H* obtaining given utility levels, and that it is productively feasible**

**- the Lagrangean is:**

$$\mathfrak{L} = U^1(x^1, G) + \sum_{h=2}^{H} \mu^h [U^h(x^h, G) - \overline{\overline{U}}^h] - \lambda F(X, G) \qquad (3)$$

**assuming specified utility levels can be reached simultaneously, the necessary conditions are:**

$$\frac{\partial \mathcal{L}}{\partial x_i^h} \equiv \mu^h \frac{\partial U^h}{\partial x_i^h} - \lambda \frac{\partial F}{\partial X_i} = 0, \; h = 1,\ldots,H \qquad (4)$$

**$\mu^h$=1 for $h$=1, (4) holding for all $i$ at the optimum,**

$$\frac{\partial \mathcal{L}}{\partial G} \equiv \sum_{h=1}^{H} \mu^h \frac{\partial U^h}{\partial G} - \lambda \frac{\partial F}{\partial G} = 0 \qquad (5)$$

**- solving (4) for $\mu^h$, and substituting into (5) and re-arranging, gives:**

$$\sum_{h=1}^{H} \frac{\dfrac{\partial U^h}{\partial G}}{\dfrac{\partial U^h}{\partial x_i^h}} = \frac{\dfrac{\partial F}{\partial G}}{\dfrac{\partial F}{\partial X_i}}, \quad i = 1,\ldots,n \qquad (6)$$

* *See the appendix*

**- each term on the left-hand side is the marginal rate of substitution between the public good and the $i$th private good for the $h$th household**

**- right-hand side is the marginal rate of transformation between the public good and private good $i$**

**(6) can be re-written as:**

$$\sum_{h=1}^{H} MRS_{Gi}^{h} = MRT_{Gi} \qquad (7)$$

**- (7) is the Samuelson rule, which compares to the rule for two private goods *i* and *j*:**

$$MRS_{ji}^{h} = MRT_{ji}, \quad all\ i,j\ and\ h \qquad (8)$$

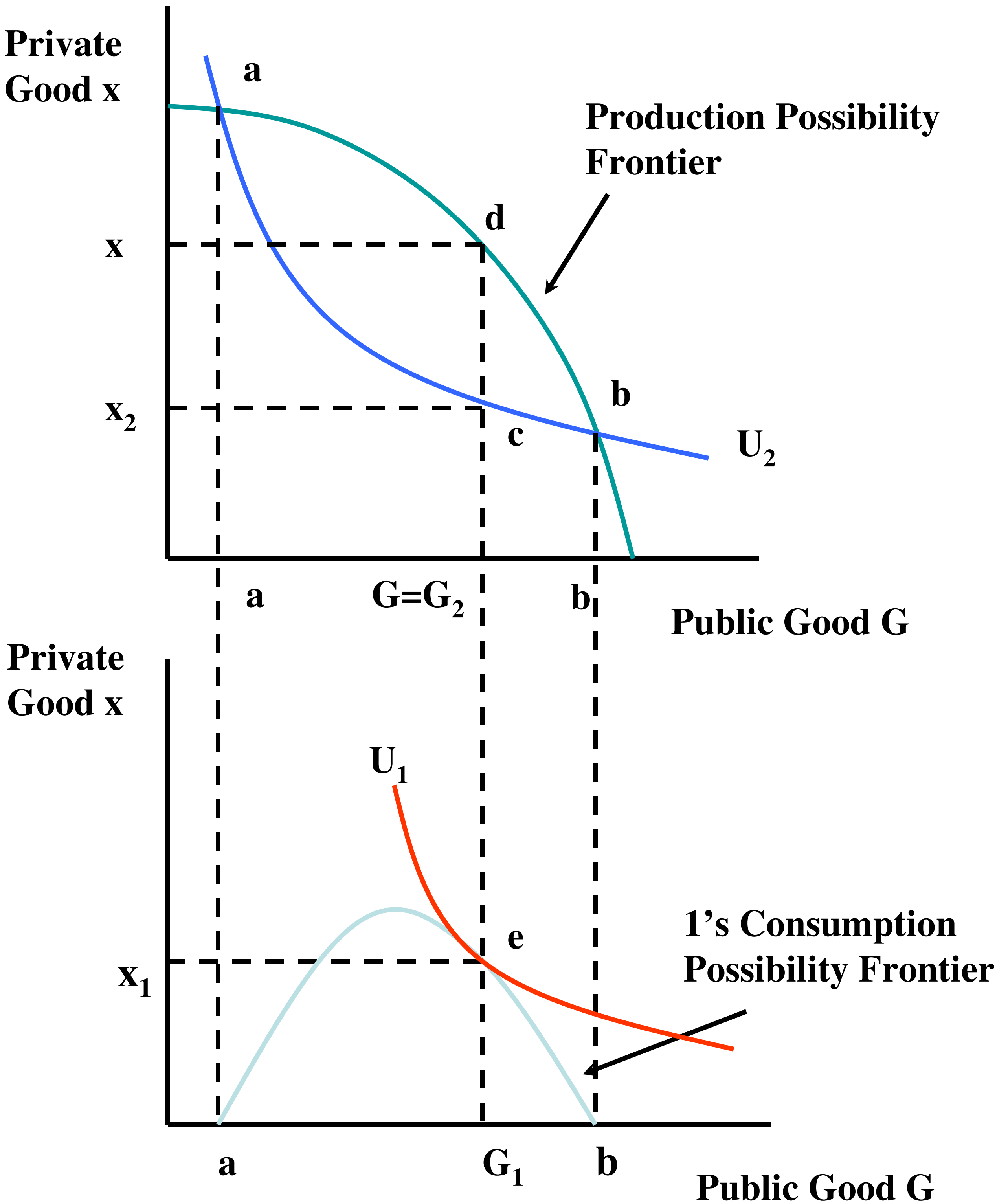
**- an extra unit of the public good increases utility of all households, so social benefit is found by summing all marginal benefits, which is then equated to marginal cost**

**- Note: non-excludability plays no role in deriving the Samuelson rule**

**- Figure 1 shows graphical presentation of Samuelson result for a two household economy with one private good and a public good**

**- trace points a and b in top part of figure to bottom part, so that each individual consumes same amount of public good between a and b**

**Figure 1: Optimal Provision of Public Good**



**- amount of private good available to households 1 and 2 varies between a and b as 2 moves along indifference curve**

**- i.e.; at a and b, 2 consumes all the private good, 1 gets none, and between a and b, 1's consumption rises then falls along consumption possibility frontier, while 2's consumption falls then rises**

**- optimal allocation for 1 is at point e, where the slope of the consumption frontier is $MRT\text{-}MRS_2$:**

$$MRS_1 = MRT - MRS_2$$

**- re-arranging:**

$$MRS_1 + MRS_2 = MRT$$

**which is equivalent to expression (7)**

**(ii) Free Disposal**

**- suppose there is free disposal of the public good, $g^h$ is consumption entering into each household's utility function, the constraint $g^h \leq G$, all $h$, being added to the maximization problem:**

$$\mathcal{L} = U^1(x^1,g^1) + \sum_{h=2}^{H} \mu^h [U^h(x^h,g^h) - \overline{U}^h]$$

$$- \lambda [F(X,G)] + \sum_{h=1}^{H} \rho^h [G - g^h] \tag{9}$$

- maximizing generates:

$$\sum_{h=1}^{H} \frac{\dfrac{\partial U^h}{\partial g^h}}{\dfrac{\partial U^h}{\partial x_i^h}} = \frac{\dfrac{\partial F}{\partial G}}{\dfrac{\partial F}{\partial X_i}} \tag{10}$$

$$where \ \frac{\partial U^h}{\partial g^h} = 0 \ if \ g^h < G$$

** See the appendix*

- marginal benefit of increasing provision of public good set equal to marginal cost, allowing for some households to be satiated so that they get no utility from extra provision

**(iii) Congestion and Optimal Provision**

- in presence of congestion, welfare of household dependent on supply of public good and use by other households:

$$U^h = U^h(x^h, g^1, \ldots, g^H, G)$$

$$\frac{\partial U^h}{\partial G} > 0,\ \frac{\partial U^h}{\partial g^h} \geq 0, and\ for\ j \neq h, \frac{\partial U^h}{\partial g^j} < 0 \qquad (11)$$

**- with no free disposal, (11) collapses to:**

$$U^h = U^h(x^h, G, H) \qquad (12)$$

**- if (11) is used, resulting necessary conditions are:**

$$\sum_{j=1}^{H} \frac{\frac{\partial U^j}{\partial g^h}}{\frac{\partial U^j}{\partial x_i^j}} = \frac{\rho^h}{\lambda \frac{\partial F}{\partial X_i}},\ i = 1, \ldots n,\ h = 1, \ldots, H \qquad (13)$$

**and:**

$$\sum_{h=1}^{H} \frac{\frac{\partial U^h}{\partial G}}{\frac{\partial U^h}{\partial x_i^h}} + \sum_{h=1}^{H} \sum_{j=1}^{H} \frac{\frac{\partial U^j}{\partial g^h}}{\frac{\partial U^j}{\partial x_i^j}} = \frac{\frac{\partial F}{\partial G}}{\frac{\partial F}{\partial X_i}},\quad i = 1, \ldots, n \qquad (14)$$

** See the appendix*

**- if $g^h < G$ for all $h$, from complementary slackness conditions, $\rho^h = 0$, and (13) states each household's use of public good should be expanded until private return is balanced by negative externalities inflicted on other households**

**- also implies second term in (14) is zero, so the standard Samuelson rule holds**

**- $g^h = G$ for some households, second term in (14) is positive, and left-hand side measures the excess benefit from increasing provision where utility is directly affected and congestion is reduced**

## ■ Lindahl Equilibrium

**- unlike the standard competitive equilibrium, Pareto efficiency will not result if households face an identical price for the public good**

**- in presence of public goods, households should face a *personalized* price, the equilibrium being known as a Lindahl (1917) equilibrium**

**(i) Simple Model**

**- single public good, where each household makes consumption decision on basis of share of cost $\tau^h$ of public good they have to pay**

**- assuming demand of $h$ for public good increases as $\tau^h \to 0$, an equilibrium is set of cost shares $\{\tau^1, ..., \tau^H\}$ that sum to 1, leading all households to demand same amount of public good**

**- equilibrium will satisfy the Samuelson rule, and be Pareto efficient**

**- economy has two households who have an endowment of $\omega^h$ units, $h$=1,2, of the numeraire (labor) which they supply inelastically**

**- single private good produced with constant returns, using numeraire, where one unit of output requires one of labor, price of private good = 1**

**- public good subject to constant returns, each unit requiring $p_G$ units of labor, so marginal rate of transformation between public and private good is a constant $p_G$**

**- household utility is:**

$$U^h = U^h(x^h, G), \; h = 1,2 \qquad (15)$$

**$x^h$ is private good, and $G$ is public good**

**- let $G^h$ be amount of public good household $h$ would like to see when faced by budget constraint:**

$$x^h + \tau^h p_G G^h = \omega^h \qquad (16)$$

**$p_G G^h$ is the total cost of providing the good, $\tau^h$ is fraction paid by $h$**

**- from (15) and (16), household $h$ chooses $G^h$ to maximize:**

$$U^h = U^h(\omega^h - \tau^h p_G G^h, G^h) \qquad (17)$$

**- necessary condition for maximization is:**

$$\frac{U_G^h}{U_x^h} = p_G \tau^h \qquad (18)$$

**- solving (18) for $G^h$ gives the Lindahl reaction function:**

$$G^h = L^h(\tau^h;\omega^h) \qquad (19)$$

**- (19) describes household's demand for a public good as a function of cost share and endowment; (19) is a decreasing function of $\tau^h$ if utility function is strictly concave** **** See the appendix***

**- Lindahl equilibrium is a pair of cost shares:**

$$\{\bar{\tau}^1,\bar{\tau}^2\} \quad such\ \ that:$$

$$(i)\ \bar{\tau}^1 + \bar{\tau}^2 = 1$$

$$and$$

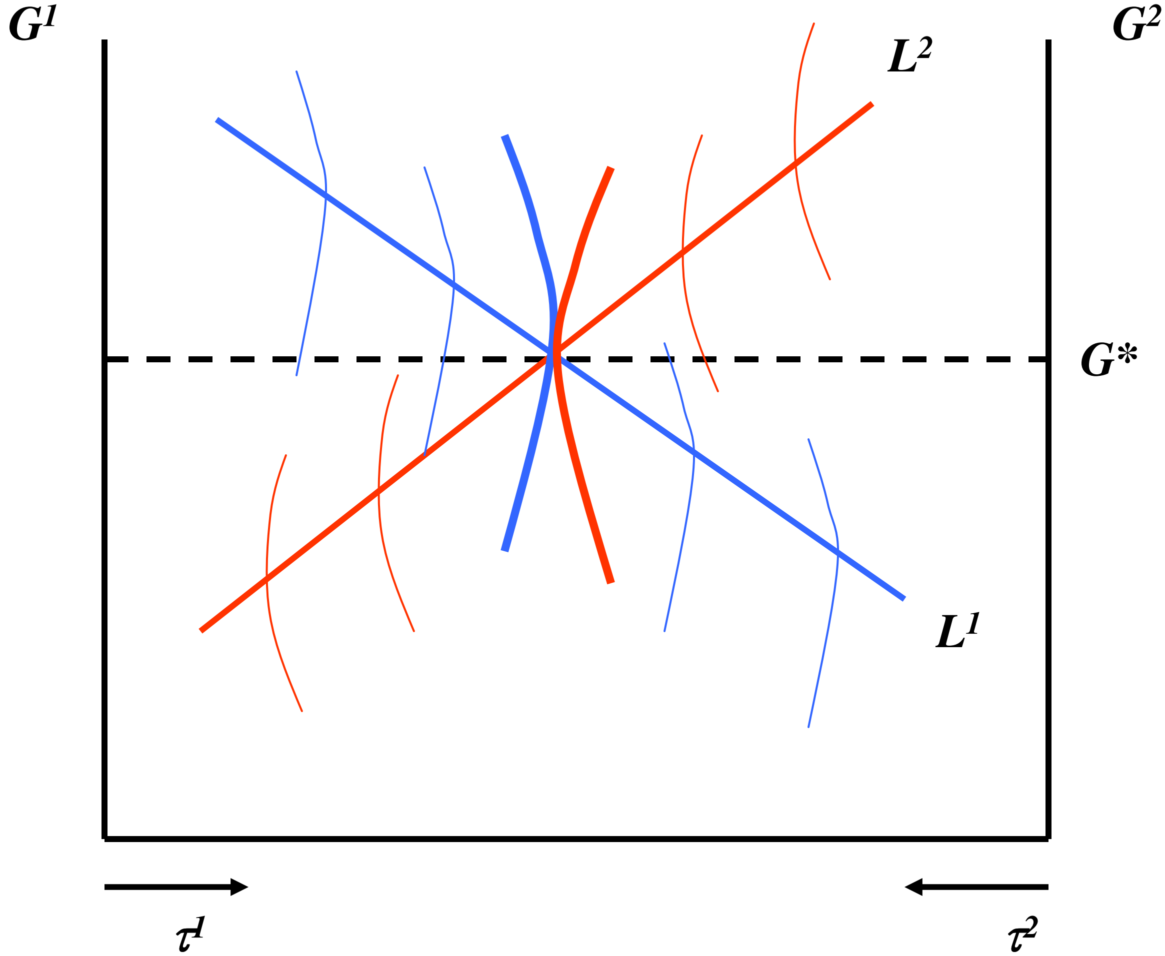
$$(ii)\ L^h(\bar{\tau}^h;\omega^h) = G^* \geq 0,\ h = 1,2$$

**- (i) guarantees enough revenue is raised to finance public good in equilibrium, (ii) ensures households are satisfied with supply**

**- Figure 2 illustrates the Lindahl equilibrium, where $L^h$, $h$=1,2 are the reaction functions, formed as loci of vertical points on the indifference curves**

**- equilibrium is at intersection of functions, where indifference curves are tangent**

**Figure 2: The Lindahl Equilibrium**



**- given (18), and summing over the households:**

$$\sum_{h=1}^{2} \frac{U_G^h}{U_x^h} = \sum_{h=1}^{2} MRS_{Gx}^h = \sum_{h=1}^{2} \tau^h p_G = p_G = MRT_{Gx} \quad (20)$$

**(20) is the Samuelson rule, so Lindahl equilibrium is Pareto efficient**

**(ii) General Model**

**- the Lindahl equilibrium can be defined as an equilibrium of a competitive economy, hence, the existence and welfare theorems of a competitive economy also apply**

**- suppose there are $s$ non-disposable public goods, and the economy's production set $Y$ is $Y \subset \Re^{n+s}$, where there are $n$ private goods, and $Y$ is a closed convex cone - i.e., constant returns**

**- a production plan is $(g,y)$, where the first $s$ elements are public goods, and the final $n$ are private goods**

**- each household $h$ possesses a consumption set $X \subset \Re^{n+s}$, with a consumption choice being $(g^h,y^h)$**

**- households have continuous preferences over $X^h$ represented by utility function $U^h(g^h, y^h)$, which is strictly monotonic and quasi-concave**

**- endowment of $h$ denoted by $\omega^h = (o, \omega^h)$, as there are no endowments of public goods**

**- a state of the economy is an array $\{g, x^1, \ldots, x^H\}$ of public and private goods, a feasible state being:**

$$(i) \quad (g; x^h) \in X^h \text{ for all } h = 1, \ldots, H$$

$$(ii) \quad \left( g; \sum_{h=1}^{H} [x^h - \omega^h] \right) \in Y$$

**($i$) ensures consumption allocation is in the consumption set, and ($ii$) ensures it is productively feasible**

**- to define a Lindahl equilibrium, let $p^h_G$ be the personalized price vector for public goods, and $p$ the common price vector for private goods**

*Lindahl equilibrium*

**- A Lindahl equilibrium with respect to the endowment, is a feasible allocation with a set of prices such that:**

$$(i)\ \left[\sum_{h=1}^{H} \bar{p}_G^h;\bar{p}\right].\left[\bar{g};\sum_{h=1}^{H}[\bar{x}^h-\omega^h]\right]\geq\left[\sum_{h=1}^{H}\bar{p}_G^h;\bar{p}\right].[g,y]$$

$$\textit{for all } (g,y)\in Y$$

$$(ii)\ \bar{p}_g^h\bar{g}+\bar{p}\bar{x}^h\leq\bar{p}\,\omega^h,$$

$$(iii)\ U^h(\bar{g};\bar{x}^h)\geq U^h(g;x^h)$$

$$\textit{for all } (g;x^h)\in X^h \textit{ such that } \bar{p}_G^h g+\bar{p}x^h\leq\bar{p}\,\omega^h$$

**- to prove existence, follow Roberts (1973) by constructing a mapping from price space for private goods and quantity space for public goods into same two spaces, and prove a fixed point exists for the mapping, and hence, equilibrium**

**- given existence of the Lindahl equilibrium, and its parallel to competitive equilibrium, versions of First and Second Theorems hold**

**- the Lindahl equilibrium is Pareto efficient (also satisfies the Samuelson rule)**

**- if a feasible allocation is Pareto efficient, there exists a price vector that will lead to the Lindahl equilibrium**

**- if the Lindahl equilibrium replicates the competitive equilibrium in presence of public goods, implies a given Pareto efficient allocation can be achieved with a suitable set of lump-sum transfers/taxation**

**- as with the Second Theorem, this is not possible if lump-sum transfers are infeasible**

**- each household in a Lindahl equilibrium faces a personalized price, which means it is in a position to falsely reveal preferences for a public good in an effort to adjust prices to its advantage**

**- such strategic behavior also undermines the Lindahl equilibrium**