

# A Term Structure Model for Agricultural Futures

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## Abstract

An extension of Schwartz's model of futures price term structure that includes seasonality is developed. The approach allows futures prices for all maturities to be estimated simultaneously by exploiting arbitrage relationships. An application to wheat futures prices is presented.

Recent practice in financial econometrics has emphasized the use of models that utilize arbitrage relationships across collections of assets. Most fully developed for fixed income securities, the entire term structure is modeled in terms of a few underlying (and possibly unobserved) factors. Recently Schwartz has applied the same approach to modeling the term structure of commodity futures prices. Schwartz also discusses the use of such models in valuing long-run commodity based assets and in evaluating investment strategies for the development of natural resource sites.

The approach has some significant potential advantages over modeling commodity futures prices using off-the-shelf time series econometric models. First, there is no need to artificially construct time series by rolling into new contracts as maturity dates are reached. Instead, all maturities are modeled simultaneously. Second, the resulting joint model for all maturities

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links each contract in a model free of arbitrage possibilities and hence the model incorporates restrictions consistent with economic equilibria. Third, due to its internal consistency, the model can be extrapolated with more confidence and thus utilized in applications beyond that of modeling futures prices. In particular, such a model can be used in evaluating investment projects that yield streams of returns over time that are linked to commodity prices.

The basic approach begins by assuming a functional form for a set of underlying state variables. Schwartz utilizes up to three such variables, a commodity spot price,  $S$ , a possibly stochastic convenience yield,  $\delta$ , and a possibly stochastic interest rate,  $r$ . The convenience yield is treated as a flow of benefits that stockholders obtain that do not accrue to holders of futures positions and, as such, is analogous to the flow of dividends to stock or coupon-bond holders.

Futures prices depend on the underlying state variables; an arbitrage relationship is exploited to determine the functional mapping between the state variables and the futures prices. In general, this relationship is defined by a partial differential equation, which, for specific choices of the state variable process, can be solved analytically.

In adapting the approach to futures on agricultural commodities, a number of extensions to Schwartz's approach may prove beneficial. In this paper we concentrate on modeling seasonality in the mean of the state variable process. Seasonality is a well documented feature of grain markets, arising from seasonal nature of the production technology. In this paper we demonstrate the basic approach and describe one way to incorporate seasonality into the model.

To review what is to come, we first discuss the state variable approach to modeling futures prices and how we model seasonality. To this end we use an extension of Schwartz's two factor model. We then apply the methodology to Chicago Board of Trade wheat futures. We finish the paper with a brief summary and discussion.

## A State Variable Approach to Modeling Futures Prices

The basic state variable approach can be summarized as follows. First, assume a stochastic process for a set of underlying state variables,  $x$ . These are modeled as a continuous time Ito diffusion process described by the stochastic differential equation<sup>1</sup>

$$dx = \mu(x, t)dt + \sigma(x, t)dz.$$

It can be shown that a futures contract whose price depends on  $x$  must satisfy an arbitrage condition that can be expressed in terms of the risk process for  $x$

$$dx = [\mu(x, t) - \sigma(x, t)\theta(x, t)]dt + \sigma(x, t)d\hat{z},$$

where  $\theta$  represents the market price of the risk in the state variables. Specifically, futures prices are martingales with respect to the risk adjusted process:

$$dF = F_x\sigma(x, t)d\hat{z}.$$

Using Ito's Lemma, this condition can be written as the partial differential equation:

$$0 = F_t(x, t; T) + F_x(x, t; T)[\mu(x, t) - \sigma(x, t)\theta(x, t)] + \frac{1}{2}\text{trace}\left(\sigma^\top(x, t)F_{xx}(x, t; T)\sigma(x, t)\right),$$

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<sup>1</sup>Hull contains a readable introduction to Ito processes and their use in finance.

subject to the boundary condition that the futures price equals the spot price at the maturity date of the futures contract,  $T$ .

Schwartz examines three specific models with one, two and three stochastic state variables. In the first, the single state variable is the spot price,  $S(t)$ ; in the general notation above  $S = x_1$ . The second adds a stochastic convenience yield,  $\delta = x_2$ . The third model adds a stochastic interest rate,  $r$  to the second model. Here we concentrate on the two factor model of Schwartz, who found it to perform significantly better than the one-factor model and as well as the three factor model when applied to oil and copper futures.

In all of these models the state variables are treated as unobservable. Although specific local cash prices may be observable, none are necessarily applicable directly to the futures market as all may incorporate some form of locational differential. Instead the spot price is treated as a limit of an instantaneous futures price:  $S(t) = x_1(t) = F(x, t, t)$ . The convenience yield is interpreted as the flow of benefits accruing to the holder of stocks and not the holder of futures (see Hull for a discussion of this interpretation).

The specific two-factor model for the spot price and the convenience yield used by Schwartz is

$$dS = (R - \delta)Sdt + \sigma_1 S dz_1$$

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2,$$

with  $dz_1 dz_2 = \rho dt$ . Here  $R$  is the total return on the holding the spot good, which consists of the rate of appreciation of the spot price plus the dividend rate  $\delta$ . A general solution for  $E[\delta]$  is

$$E_\tau[\delta_t] = m(t) + e^{-\kappa\Delta t} (\delta_\tau - m(\tau)),$$

where

$$m(t) = \kappa e^{-\kappa t} \int^t e^{\kappa \tau} \alpha(\tau) d\tau.$$

The function  $m(t)$  can be interpreted as the long-run mean towards which the system tends after being perturbed.

The futures, however, depends of the risk adjusted process . This requires a bit of explanation. First, the spot price, although not observable, is treated as an asset and hence must satisfy an arbitrage condition:

$$R = \mu + \delta = r + \sigma\theta,$$

that the total rate of return on the asset,  $R$ , which is identically equal to its rate of capital appreciation,  $\mu$ , plus its dividend rate,  $\delta$ , must equal the risk-free rate of interest,  $r$ , plus a risk premium equal to the volatility of the process,  $\sigma$ , times the market price of risk,  $\theta$ . This allows the risk adjusted drift for the spot price,  $\mu - \sigma\theta$ , to be replaced by  $r - \delta$ .

The associated futures price satisfies

$$F_T = (r - \delta)SF_S + (\kappa(\alpha - \delta) - \lambda)F_\delta + \frac{1}{2}\sigma_1^2 S^2 F_{SS} + \rho\sigma_1\sigma_2 SF_{S\delta} + \frac{1}{2}\sigma_2^2 F_{\delta\delta},$$

with the boundary condition  $F(S, \delta, 0) = S$  and where  $r$  is the risk-free rate of interest. It is straightforward to verify that

$$F(S, \delta, T) = S \exp \left( A(T) - \delta \frac{1 - e^{-\kappa T}}{\kappa} \right)$$

solves the PDE for futures prices for some function  $A(t)$  and that this function must satisfy

$$A'(t) = (\kappa\alpha - \lambda + \rho\sigma_1\sigma_2) \frac{1 - e^{-\kappa(T-t)}}{\kappa} - \frac{\sigma_2^2}{2} \left( \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2 - r,$$

with  $A(T) = 0$ .

Schwartz treats all of the model parameters as constants but the analysis is unchanged if the parameters are treated as functions of time. Seasonal effects can therefore be introduced into the model by making the parameters be periodic functions of time, with a periodicity of one year. Furthermore, any seasonal variation in the model parameters (other than  $\kappa$ ) influences the futures price only through the  $A(t)$  term.

A simple way to introduce seasonality into this model is through the parameter  $\alpha$ . This makes  $\delta$  be mean-reverting to a seasonal function rather than to a constant value. Specifically,  $\alpha(t)$  is modeled as a truncated Fourier series, i.e.,

$$\alpha(t) = \eta_0 + \sum_i \cos(\nu_i t) \eta_i + \sin(\nu_i t) \theta_i. \quad (1)$$

where  $\nu_i = 2\pi i$ .

When seasonality enters through  $\alpha(t)$ , it is easily verified that  $A(t)$  is given by

$$\begin{aligned} A(t) = & \left( r + \frac{\lambda}{\kappa} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa} \right) (T-t) + \frac{\sigma_2^2}{4} \frac{1 - e^{-2\kappa(T-t)}}{\kappa^3} \\ & - \left( \frac{\lambda}{\kappa} + \rho\sigma_1\sigma_2 - \frac{\sigma_2^2}{\kappa} \right) \frac{1 - e^{-\kappa(T-t)}}{\kappa^2} \\ & + \int_t^T (1 - e^{-\kappa(T-\tau)}) \alpha(\tau) d\tau \end{aligned}$$

The last term in this expression can be factored as

$$\int_t^T (1 - e^{-\kappa(T-\tau)}) \alpha(\tau) d\tau = \int_t^T \alpha(\tau) d\tau - \frac{1}{\kappa} (m(T) - e^{-\kappa(T-t)} m(t)).$$

When  $\alpha(t)$  is specified as a truncated Fourier series (eq. 1), this can be written as

$$\begin{aligned} \int_t^T (1 - e^{-\kappa(T-\tau)}) \alpha(\tau) d\tau = & \\ & \eta_0(T-t) + \sum_i \left( [\sin(\nu_i T) - \sin(\nu_i t)] \frac{\eta_i}{\nu_i} - [\cos(\nu_i T) - \cos(\nu_i t)] \frac{\theta_i}{\nu_i} \right) \\ & - \frac{1}{\kappa} (m(T) - e^{-\kappa(T-t)} m(t)). \end{aligned}$$

When  $\alpha$  is a constant this simplifies to

$$\left( (T - t) - \frac{1 - e^{-\kappa(T-t)}}{\kappa} \right) \alpha,$$

the expression in Schwartz.

## An Application To Wheat Futures

Friday futures prices for the Chicago Board of Trade wheat futures contract were used to estimate parameters of the term structure model presented above. The data set spanned the period from July 1, 1975 to March 10, 1998. All futures contracts with maturities of one year or less were used, excluding contracts in their delivery month, with a total of 5583 price observations.

Maximum likelihood estimation was used, with the likelihood computed via the Kalman filter, as described by Schwartz. Like Schwartz, we used the Euler discretization of the state transition equation, rather than computing the exact discrete time mean and variance. Due to identification problems, it was not possible to obtain useful estimates of  $\lambda$ . This parameter, representing the market price of convenience yield risk is expected to be small for wheat and hence was set to zero. A value of zero was used for the interest rate. As noted by Schwartz, the specific value chosen does not affect model fit but only how certain parameters are interpreted. Specifically, convenience yield is interpreted as the total flow of payments resulting from holding stocks, including the costs (negative payments) of interest, storage charges, insurance, etc.

Parameter estimates are presented in Table 1 for four alternative models with differing numbers of seasonal terms. The results provide clear evidence for the importance of the

seasonality terms in the convenience yield. The seasonal terms (the  $\eta_i$  and  $\theta_i$ ) all have coefficients that are significantly different from zero. Using the Akaike information criteria, however, the order 2 model (with four seasonality coefficients) is judged to have the appropriate balance between fit and parsimony. The four instantaneous mean functions for the convenience yields are plotted in Figure 1. The double-humped function has two troughs, one in July and the other in December or January. These correspond to the harvest periods for winter and spring wheat, when relative abundance generally causes the convenience yield to be low.<sup>2</sup>

Turning to other parameter values, the value for  $\alpha$  of 0.0019 can be interpreted as the long run average level of convenience yield, suggesting that the convenience yield is generally quite small (two tenths of a percent of the value of wheat). The size of the instantaneous variance on the convenience yield ( $\sigma_2 = 0.4527$ ) together with the seasonal shifts from -0.6 to 0.9, indicate a high degree of variability in convenience yield over time. The mean reversion parameter,  $\kappa = 1.8228$ , indicates a fairly rapid rate of reversion, with a half life of between four and five months.

The total return on the holding spot goods,  $R = 0.0277$ , indicates that the margins on storage of wheat are rather small, 3%, above the risk-free rate. This is true in spite of the fact that the risks are rather large; the annualized volatility in the spot price is  $\sigma_1 = 0.2326$ . Finally, the correlation between the spot price and the convenience yield,  $\rho = 0.3377$ , reflects a positive relationship, which is consistent with the stylized fact that convenience yield, like price, is low (high) when stocks are plentiful (scarce).

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<sup>2</sup>The CBOT contract is on spring wheat but winter and spring wheat exhibit a high degree of substitutability in demand.

## Conclusions

Schwartz has provided an integrated framework for modeling futures prices that incorporates arbitrage relationships and allows futures of all maturities to be used simultaneously to estimate model parameters. The approach avoids some of the problems that arise in trying to get futures prices to fit the assumptions of off-the-shelf time series models. The most obvious of these is the roll-over problem that arises at a contract's maturity. More subtle and often simply ignored are the complex relationships that arise due to the maturity structure of futures and the convergence of futures to spot prices at maturity. These effects are incorporated in the model structure in Schwartz's approach and hence do not require special attention.

One important extension to Schwartz's model, especially for agricultural commodities, is the incorporation of seasonal effects into the model. All model parameters can be made to be seasonal functions of time. With the exception of the mean reversion speed parameter, this extension only changes the definition of the  $A(t)$  term in the futures price function. The specific seasonal function used here allowed the mean of the convenience yield to vary over the year. In an application to wheat futures, this was clearly warranted by the results presented.

## References

- Hull, John. *Options, Futures and Other Derivatives*, 4th ed. Prentice Hall. 1999.
- Schwartz, E.S. "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging." *Journal of Finance* 52 (1997): 923-973.

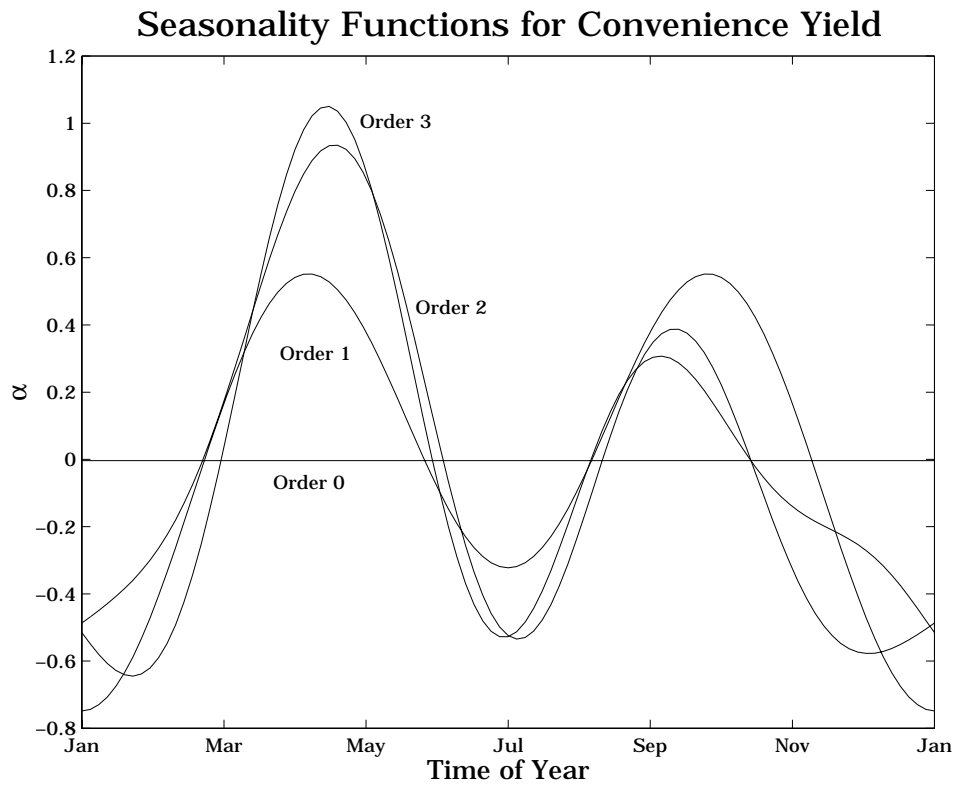


Figure 1

Table 1. Parameter Estimates for Wheat Futures

	Order 0			Order 1			Order 2			Order 3		
	MLE	StdErr	t	MLE	StdErr	t	MLE	StdErr	t	MLE	StdErr	t
$\mu$	0.0002	0.0499	0.0043	0.0218	0.0523	0.4175	0.0277	0.0530	0.5217	0.0274	0.0532	0.5154
$\alpha$	-0.0035	0.0075	-0.4702	0.0031	0.0039	0.7955	0.0019	0.0041	0.4717	0.0021	0.0042	0.5027
$\kappa$	1.0436	0.0598	17.4538	1.7576	0.0342	51.3213	1.8228	0.0327	55.7004	1.8234	0.0322	56.6254
$\sigma_1$	0.2187	0.0071	31.0258	0.2301	0.0054	42.5796	0.2326	0.0054	43.3371	0.2326	0.0055	42.6358
$\sigma_2$	0.2434	0.0195	12.4994	0.4059	0.0206	19.6571	0.4527	0.0231	19.6308	0.4495	0.0237	18.9373
$\sigma_\epsilon$	0.0233	0.0002	154.4327	0.0156	0.0001	119.9569	0.0149	0.0001	131.4274	0.0148	0.0001	132.6246
$\rho$	0.3980	0.0632	6.3015	0.3428	0.0431	7.9472	0.3377	0.0396	8.5369	0.3372	0.0404	8.3568
$\eta_1$				-0.2129	0.0123	-17.3199	-0.2088	0.0113	-18.4122	-0.2065	0.0118	-17.4411
$\theta_1$				-0.5385	0.0120	-44.8202	-0.5074	0.0130	-39.0735	-0.5234	0.0134	-38.9702
$\eta_2$							0.2274	0.0376	6.0539	0.2127	0.0379	5.6156
$\theta_2$							0.2879	0.0473	6.0864	0.2119	0.0524	4.0430
$\eta_3$										-0.1102	0.0485	-2.2728
$\theta_3$										-0.1839	0.0472	-3.8998
MSE		7.5757			4.9371			4.677			4.6481	
ll		-13.7181			-15.0226			-15.126			-15.149	

5583 Observations.